# Rational Expectations Equilibrium with an Automated Market Maker

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**Abstract**

We characterize the Grossman–Stiglitz rational expectations equilibrium with costly information acquisition in a prediction market operated by an automated market maker. The market maker executes all orders in a single batch under the logarithmic market scoring rule, and uninformed traders condition their beliefs on the execution price. In a sufficiently liquid market, greater liquidity makes the price more informative only by inducing more traders to acquire the signal: it damps price movements toward zero or one and thereby raises the signal's value. Noise trading shelters informed traders only up to a point: at any finite liquidity, the signal's value vanishes at both extremes of noise-trading volatility. The market maker's expected cash flow is positive in a shallow market and negative in a deep one.

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# 1. Introduction

The growth of prediction markets has renewed interest in their ability to aggregate information about uncertain events.[1] Their value as forecasting institutions also bears on the regulatory debate over the economic purposes of event contracts.[2] The information reflected in prices, however, depends on traders' incentives to acquire it. When prices reveal private information, traders who have not acquired it can use that information in their own trading decisions, reducing the informational advantage that makes acquisition profitable. Price informativeness and information acquisition must therefore be studied jointly. We study this interaction in a Grossman–Stiglitz rational expectations model of a prediction market operated by an automated market maker, characterizing the equilibrium acquisition of information, its revelation through prices, and the market maker's expected trading profit.

A prediction market provides a natural setting for studying the interaction between information acquisition and price revelation identified by Grossman and Stiglitz (1980) (GS). A binary contract's price is commonly interpreted as an event probability and can be evaluated against the realized outcome.[3] This potential forecasting value may go unrealized when trading is thin, as it is for many contracts on Polymarket and Kalshi (Chetia, 2026), both of which operate order books. An automated market maker can provide a counterparty in these thin markets by committing liquidity before information is acquired. We examine how this liquidity provision affects acquisition incentives and price informativeness.

We study the logarithmic market scoring rule (LMSR) of Hanson (2003), which has been

[1] Combined monthly notional trading volume on Kalshi and Polymarket increased from approximately $26 billion in May 2026 to $53 billion in July 2026 (Radde, 2026).

[2] The Commodity Futures Trading Commission's proposed public-interest assessment recognizes information aggregation and forecasting as potential benefits of event contracts, alongside hedging and price discovery (Commodity Futures Trading Commission, 2026, pp. 35829–35830), and weighs these benefits against the risks of manipulation, compromised settlement integrity, and misuse of nonpublic information.

[3] A stock price, by contrast, reflects both expected cash flows and discount rates, which cannot be separately identified from the price alone.

deployed in practice (Othman et al., 2013; Gnosis, 2020). Its liquidity parameter determines both price sensitivity and the market maker's maximum loss, making the capital required to supply liquidity explicit. Its logarithmic form facilitates the analysis of costly information acquisition and learning from prices in rational expectations equilibrium.[4]

In GS's rational expectations equilibrium, full revelation of costly information would eliminate the incentive to acquire it. Noise trading limits price revelation and allows informed traders to earn a return on their information. In our model, an automated market maker supplies liquidity under the LMSR. A continuum of risk-averse traders decide whether to acquire a common signal at a cost and submit demands for a binary event contract. All orders, including an exogenous noise order, execute in a single batch at a common price: the average price per share charged by the LMSR for the aggregate net order.[5] This execution price is publicly observed, and uninformed traders condition their demands on the information it conveys.

The model distinguishes the execution price from the public posterior, the probability of the event conditional on that price. The execution price determines trading payments, while the public posterior reflects the information revealed through trading; no liquidity level makes the two coincide in every state. To our knowledge, this paper provides the first characterization of a GS rational expectations equilibrium with an automated market maker.

We first characterize price revelation, taking information acquisition as given. For any fixed fraction of informed traders, a continuous equilibrium exists and its price map is unique. The execution price reveals order flow—the sum of the informed traders' aggregate order at the opening quote and the noise order—but does not distinguish its informed and noise

[4]Under the LMSR, changes in the log odds of the marginal quote are linear in net order flow. Constant-product rules do not share this linear relation.

[5]Orders are executed in common proportion, yielding the Aumann–Shapley price described in Section 2.2. Total net payments equal the change in the LMSR cost function, so the market maker retains the LMSR's bound on its worst-case loss after settlement.

components. Greater liquidity reduces the price movement associated with a given order flow without changing the posterior that uninformed traders infer from it. Price informativeness is therefore independent of liquidity at a fixed informed fraction; liquidity affects it only through traders' information acquisition decisions.

The signal's private value comes from the information that the execution price does not reveal. The value of this residual information depends on its precision and on the execution price: a price near zero or one leaves little to gain from being informed. When informed and uninformed traders coexist in equilibrium, the value of the signal equals its cost.

We then characterize how liquidity and noise-trading volatility affect information acquisition and price informativeness. Informed traders' orders tend to move the price toward zero or one, reducing the value of the signal. In a sufficiently liquid market, greater liquidity dampens these price movements and raises the signal's value. At an interior acquisition equilibrium, a larger fraction of traders therefore choose to become informed, making the price more informative.

Noise trading affects acquisition through both price revelation and the execution price. At any finite liquidity and a fixed positive fraction of informed traders, the signal's value vanishes at both extremes of noise-trading volatility. With negligible noise, the price reveals the signal. With extreme noise, the execution price approaches zero or one, leaving little to gain from acquiring the signal even though it remains largely private. The signal's value therefore reaches its maximum at an intermediate volatility. This reversal arises from the combination of a binary payoff and the finite liquidity the automated market maker posts: unlike in GS, noise can reduce the signal's value by moving the execution price toward the payoff bounds. As liquidity tends to infinity, however, the execution price approaches the prior, eliminating this adverse price effect.

In a sufficiently liquid market, greater noise-trading volatility raises the signal's value at

a fixed informed fraction by reducing price revelation. At an interior acquisition equilibrium, this increase induces more traders to acquire information. Greater noise therefore has two opposing effects on price informativeness: it obscures existing information but encourages its acquisition.

In the infinite-liquidity limit, these effects exactly offset each other, provided the acquisition equilibrium remains interior. With other primitives fixed, the acquisition cost determines how much of the signal must remain private to compensate informed traders. An increase in noise-trading volatility therefore induces a proportional increase in the informed fraction, preserving the signal-to-noise ratio in order flow and leaving price informativeness unchanged.

We next examine the financial consequences of supplying liquidity. At a fixed positive fraction of informed traders, the market maker earns an expected profit in a sufficiently shallow market and incurs an expected loss in a sufficiently deep one. This difference reflects the gap between the execution price and the public posterior. In the low-liquidity limit, the execution price overshoots the public posterior: the market maker sells above it and buys below it. Noise traders lose in expectation because their orders move the price against them; their losses cover the informed and uninformed traders' expected trading profits and leave a profit for the market maker. In the infinite-liquidity limit, the execution price approaches the prior and noise traders' expected losses vanish. The market maker then finances the expected trading profits of both informed traders and uninformed traders who learn from the price.

Liquidity also affects the accuracy of the execution price when interpreted as a probability forecast. At a fixed positive fraction of informed traders, greater liquidity initially improves accuracy by reducing the price's overshoot of the public posterior. At high liquidity, however, it draws the price toward the prior and reduces accuracy. Forecast accuracy is therefore

maximized at an intermediate liquidity.

Finally, we let the market maker choose liquidity before trading. It values the forecast accuracy of the execution price, bears expected settlement payouts net of trading receipts, and pays a capital charge on reserves covering its maximum possible loss. In a sufficiently liquid market with an interior acquisition equilibrium, greater liquidity induces more traders to become informed but also draws the execution price toward the prior. The latter effect dominates: the price becomes more informative yet less accurate as a forecast, while the market maker's expected loss increases.

Section 2 presents the model. Section 3 characterizes the equilibrium at a fixed fraction of informed traders, derives the value of information, and examines the effects of liquidity and noise-trading volatility. Section 4 characterizes the equilibrium fraction of informed traders. Section 5 examines the market maker's expected loss and the accuracy of the execution price as a forecast. Section 6 studies the market maker's choice of liquidity. Section 7 concludes.

## 1.1 Related Literature

The paper contributes to the noisy rational-expectations literature on information acquisition. GS establish the tension between costly information acquisition and revelation through prices. Hellwig (1980) studies information aggregation with a continuum of traders, and Verrecchia (1982) characterizes traders' optimal information acquisition when prices partially reveal private information. Breon-Drish (2015) establishes existence and uniqueness of equilibrium beyond the linear-Gaussian setting, while Albagli et al. (2024) show how payoff asymmetry shapes the wedge between the price and the expected payoff. We study information acquisition and price revelation in a market served by an automated market maker that commits liquidity before traders acquire information. We characterize equilibrium both with a fixed fraction of informed traders and with endogenous information acquisition, show-

ing how the market maker's liquidity affects acquisition incentives and, through them, price informativeness.

A second strand studies information aggregation in prediction markets (see Wolfers and Zitzewitz, 2004, for a survey). Hanson (2003) introduces market scoring rules, and Ostrovsky (2012) establishes information aggregation for separable securities. Ottaviani and Sørensen (2015) show how heterogeneous beliefs and wealth effects lead prices to underreact to information. We study how information acquisition responds to the liquidity supplied by an automated market maker. Liu and Pang (2026) also endogenize acquisition under the LMSR, but in a sequential-trading model in which an arbitrageur learns from an order without observing whether its sender is informed. Here, a continuum of traders decide whether to become informed, and uninformed traders learn from the price at which they trade. Our focus is the joint determination of information acquisition and price informativeness in rational expectations equilibrium.

A third strand studies liquidity provision under asymmetric information and its extension to automated market makers. In Kyle (1985) and Glosten and Milgrom (1985), competitive market makers infer asset values from orders and set prices to earn zero expected profits conditional on the information they observe. In decentralized finance, liquidity providers commit to a pricing rule and face arbitrage losses as external prices change (Capponi and Jia, 2025; Lehar and Parlour, 2025). Our market maker likewise commits to a pricing rule, but the information revealed through trading is itself an object of value. We examine how liquidity affects information acquisition, learning from prices, and the market maker's expected trading profit. The market maker chooses liquidity to balance the value of forecast accuracy against its expected loss and the capital charge on its reserves.

# 2. Model setup

A market maker opens a market on a binary event using the logarithmic market scoring rule (LMSR) of Hanson (2003) and commits to its liquidity parameter before trading. A continuum of traders decide whether to acquire a costly signal about the event and then trade with the market maker in a single batch.

## 2.1 Primitives

Let $B \in \{0, 1\}$ indicate whether the event occurs. Agents share a common prior $\delta_0 = \Pr(B = 1) \in (0, 1)$, which is also the opening price of the YES contract. Throughout, log denotes the natural logarithm, and $\operatorname{logit} x := \log \frac{x}{1-x}$ denotes the log-odds of $x \in (0, 1)$. We write $\sigma(z) := 1/(1 + e^{-z})$ for the inverse of the log-odds function and $\mathscr{A}(\eta) := \log(1 + e^{\eta})$. For $p, q \in (0, 1)$, the Bernoulli relative entropy is $\mathrm{KL}(p \,\|\, q) := p \log \frac{p}{q} + (1 - p) \log \frac{1-p}{1-q}$. We use the identities $\sigma(\operatorname{logit} p) = p$, $\mathscr{A}(\operatorname{logit} p) = -\log(1 - p)$, and $\mathscr{A}' = \sigma$.

A unit mass of ex ante identical traders have deterministic initial wealth $W$ and utility $-e^{-w/\tau}$ over terminal wealth $w$, with constant absolute risk tolerance $\tau > 0$. Each trader may acquire a common signal $S$ about the event at a fixed monetary cost $c_I > 0$; a fraction $\vartheta \in [0, 1]$ of traders do so. Informed traders share the posterior $P_I := \Pr(B = 1 \mid S)$, and the signal's log-likelihood ratio is

$$L := \operatorname{logit} P_I - \operatorname{logit} \delta_0. \tag{1}$$

Thus, $P_I = \sigma(\operatorname{logit} \delta_0 + L)$. An exogenous noise order $\xi$ is a net quantity of YES shares, independent of $(B, S)$, with a density and a distribution that does not depend on liquidity or the fraction of informed traders. After trading, each YES share pays one dollar if $B = 1$

and zero otherwise.

We specify the signal and noise distributions as follows.

**Assumption 1** (Gaussian benchmark)**.** *The signal's log-likelihood ratio satisfies* $L \mid B = 1 \sim N(\bar{\rho}^2/2, \bar{\rho}^2)$ *and* $L \mid B = 0 \sim N(-\bar{\rho}^2/2, \bar{\rho}^2)$*, where* $\bar{\rho} > 0$ *measures signal precision. The noise order satisfies* $\xi \sim N(0, \sigma_\xi^2)$*, where* $\sigma_\xi > 0$*.*

This specification can be interpreted as observing $Y = (2B-1)\bar{\rho}/2 + \varepsilon$, where $\varepsilon$ is standard normal noise and independent of $B$. Bayes' rule gives the log-likelihood ratio $L = \bar{\rho}Y$, which has the conditional distributions in Assumption 1. A higher $\bar{\rho}$ makes the signal more informative by increasing the distance between the two conditional means of $Y$ relative to its conditional standard deviation.

The market maker first commits to liquidity, after which information acquisition and trading follow the timing of Grossman and Stiglitz (1980). At $t = 0$, the market maker posts the LMSR with liquidity $b > 0$. At $t = 1$, traders decide whether to acquire the signal. At $t = 2$, informed traders observe $S$, and the noise order $\xi$ is realized. At $t = 3$, all orders are executed in a single batch. At $t = 4$, the event outcome $B$ is realized and positions settle.

## 2.2 Market making and batch execution

For liquidity $b > 0$, the market maker posts the LMSR cost function and marginal quote

$$C_b(Q) = b\,\mathrm{sA}(Q/b), \qquad C_b'(Q) = \sigma(Q/b), \tag{2}$$

where $Q$, the market maker's inventory, is the number of outstanding net YES shares.[6] The market opens at $Q = b \operatorname{logit} \delta_0$. A net order of $b(\operatorname{logit}\delta - \operatorname{logit}\delta_0)$ YES shares moves the

[6]With outstanding YES and NO shares $q_1, q_0$, LMSR specifies $C(q_1, q_0) = b\log(e^{q_1/b} + e^{q_0/b})$. Buying one share of each costs one dollar and pays one dollar at settlement, so we use $Q = q_1 - q_0$ and $C_b(Q) = C(Q, 0)$.

quote from $\delta_0$ to $\delta$.

Trading takes place in a single batch. Each trader submits a demand schedule specifying a net YES order at each possible execution price. The noise order $\xi$ is random but does not respond to price. The market maker executes all orders together, completing the same fraction of every order at any point during execution. Each additional share trades at the marginal quote corresponding to the net quantity executed so far. Every trader faces the same execution price: the average marginal quote over this proportional execution.

**Definition 1** (The pro-rata batch)**.** *Given demand schedules $x_i(\cdot)$ and a noise order $\xi$, the batch clears at an execution price $\bar{\delta} \in (0,1)$ with total net order $x \in \mathbb{R}$ if:*

*(i) The market maker fills all orders proportionally, charging the marginal quote $C_b'(b \operatorname{logit} \delta_0 + sx)$ when the fraction of each order filled is $s \in [0,1]$.*

*(ii) All orders, including the noise order, trade at the common price per YES share*

$$\bar{\delta} = \int_0^1 C_b'(b \operatorname{logit} \delta_0 + sx)\, ds. \tag{3}$$

*(iii) The total net order satisfies $x = \int x_i(\bar{\delta})\, di + \xi$.*

*The terminal quote is $\delta^* := \sigma(\operatorname{logit} \delta_0 + x/b)$. Trading ends after execution, and positions settle when $B$ is realized.*

The batch operates as a call market in which the LMSR cost function determines the common execution price. For each YES share traded, buyers pay the execution price to the market maker and sellers receive it from the market maker. The market maker publishes the execution price. The terminal quote is the marginal quote corresponding to the market maker's inventory after execution. The next lemma characterizes the execution price and the market maker's net cash flow, and bounds its loss.

**Lemma 1.** *Fix $b > 0$, $\delta_0 \in (0,1)$, and a total net order $x \in \mathbb{R}$, with terminal quote $\delta^*$ and cost increment $\Delta C := C_b(b \operatorname{logit} \delta_0 + x) - C_b(b \operatorname{logit} \delta_0)$.*

*(i) The execution price satisfies*

$$\bar{\delta} = \begin{cases} \dfrac{\mathscr{A}(\operatorname{logit} \delta^*) - \mathscr{A}(\operatorname{logit} \delta_0)}{\operatorname{logit} \delta^* - \operatorname{logit} \delta_0}, & x \neq 0, \\ \delta_0, & x = 0. \end{cases} \tag{4}$$

*(ii) The market maker's net receipts are $x\bar{\delta} = \Delta C$, and its net cash flow after settlement is $\Delta C - xB$.*

*(iii) If $x \neq 0$, $\bar{\delta}$ lies strictly between $\delta_0$ and $\delta^*$. As a function of $\delta^*$, $\bar{\delta}$ is smooth and strictly increasing on $(0,1)$.*

*(iv) In every state, $xB - \Delta C \leq b\, M(\delta_0)$, where $M(\delta_0) := \max\{-\log \delta_0, -\log(1-\delta_0)\}$.*

*Proof.* Appendix A.1.

The market maker receives net payments equal to the LMSR cost increment $\Delta C$ from all traders, including the noise traders, and makes a net settlement payment of $xB$ to them. Buyers and sellers face the same execution price, so there is no bid–ask spread within the batch. Proportional execution preserves the LMSR loss bound.

The execution price follows the Aumann–Shapley rule for allocating costs (Aumann and Shapley, 1974). This rule assigns each order a per-share price equal to its average marginal cost as all orders are filled in the same proportion. Because the LMSR cost increment depends only on the total net order, all orders face the same marginal cost per share at each stage of execution and hence the common execution price in (4). Billera and Heath

(1982) and Mirman and Tauman (1982) characterize Aumann–Shapley prices axiomatically for nonnegative demands.[7]

Our assumption of batch execution at a common price is also consistent with the practices of some decentralized exchanges.[8] In our model, proportional execution determines the common price, but a rational expectations equilibrium also exists if all orders execute at the terminal quote.[9]

## 2.3 Trader demand and market clearing

A trader who submits a net YES order $x$ at the execution price $\bar{\delta}$ has terminal wealth $W + x(B - \bar{\delta})$, less the acquisition cost if she is informed. The acquisition cost is sunk at the time of trading and, with constant absolute risk tolerance, does not affect her optimal order. The next lemma characterizes the optimal order.

**Lemma 2.** *For a trader with posterior $P \in (0,1)$ who takes the execution price $\bar{\delta} \in (0,1)$ as given, the unique optimal net YES order is*

$$x(P, \bar{\delta}) = \tau(\operatorname{logit} P - \operatorname{logit} \bar{\delta}). \tag{5}$$

*Her certainty-equivalent gain from trading is $\tau \operatorname{KL}(\bar{\delta} \,\|\, P)$.*

*Proof.* Appendix A.2.

[7] Orders in our market may be positive or negative. We apply the pricing formula in (4) to both purchases and sales.

[8] CoW Protocol settles each batch of orders at uniform clearing prices, and Penumbra executes swaps in a batch and allocates the assets received among traders in proportion to the amounts they contribute; see "MEV Protection," CoW Protocol Documentation, `https://docs.cow.fi/cow-protocol/concepts/benefits/mev-protection`, and "Batch Swaps," The Penumbra Protocol, `https://protocol.penumbra.zone/main/dex/swap.html`.

[9] Under terminal-quote pricing, each trader's demand retains the form of (5) at the price she pays, the price reveals the same statistic, and we can solve explicitly for the equilibrium price. The alternative model is available upon request.

A trader's net YES order equals her risk tolerance times the difference in log-odds between her posterior and the execution price. Informed traders use the posterior $P_I$, whereas uninformed traders infer the event probability from the execution price, forming the posterior $P_U := \Pr(B = 1 \mid \bar{\delta})$.

The schedules of Lemma 2 determine how the batch clears at the traders' posteriors.

**Lemma 3.** *Fix traders' posteriors $P_i \in (0,1)$ with $\int \operatorname{logit} P_i \, di$ finite, and let each trader submit the demand schedule $x(P_i, \cdot)$ in (5). For every noise order $\xi \in \mathbb{R}$, a unique pair $(\bar{\delta}, x) \in (0,1) \times \mathbb{R}$ clears the batch. This pair depends on the posteriors and the noise order only through*

$$\int \operatorname{logit} P_i \, di + \xi/\tau, \tag{6}$$

*and both the execution price and the terminal quote are strictly increasing in this statistic.*

*Proof.* Appendix A.3.

Holding traders' posteriors fixed, a higher execution price reduces their net demand and increases the net quantity supplied by the market maker. More optimistic beliefs and a larger noise order both raise the execution price. The price alone does not distinguish their contributions.

# 3. Equilibrium at a fixed informed fraction

We now fix the fraction $\vartheta \in [0,1]$ of informed traders. Uninformed traders form their common posterior from the execution price. The price clears the batch given traders' optimal demand schedules. We first define equilibrium and then characterize how the signal and noise affect prices and order flow in the Gaussian benchmark.

The state of the market is described by $(L, \xi)$, the signal's log-likelihood ratio and the noise order. An equilibrium specifies the execution price in each state and uninformed traders' posterior belief at each possible execution price.

**Definition 2** (Equilibrium at a fixed informed fraction)**.** *Fix $\vartheta \in [0,1]$. An* equilibrium *consists of a measurable execution-price function $\bar{\delta} : \mathbb{R}^2 \to (0,1)$ and a measurable posterior function $P_U : (0,1) \to (0,1)$ for uninformed traders, satisfying:*

*(i)* Inference. *$P_U(p) = \Pr(B = 1 \mid \bar{\delta} = p)$ for almost every execution price $p$ under the distribution of $\bar{\delta}$.*

*(ii)* Optimality. *At each possible execution price $p \in (0,1)$, informed and uninformed traders submit orders $x(P_I, p)$ and $x(P_U(p), p)$, respectively, as given by* (5)*.*

*(iii)* Clearing. *In every state $(L, \xi)$, the execution price $p = \bar{\delta}(L, \xi)$ and total net order $x$ satisfy*

$$\begin{aligned} x &= \tau[\vartheta \operatorname{logit} P_I + (1-\vartheta) \operatorname{logit} P_U(p) - \operatorname{logit} p] + \xi, \\ p &= \int_0^1 C_b'(b \operatorname{logit} \delta_0 + sx)\, ds. \end{aligned} \tag{7}$$

*An equilibrium is* continuous *if $\bar{\delta}$ is continuous on $\mathbb{R}^2$.*

The definition describes a rational expectations equilibrium with price-taking traders and an automated market maker. The execution price depends on the signal and the noise order. The noise order is independent of $(B, S)$, so the execution price provides no additional information about the event to a trader who observes the signal. The informed trader's posterior in condition (ii) therefore remains $P_I$ (Lemma 5 in Appendix A.4).

The next lemma shows that, in every equilibrium, the execution price reveals the informed

traders' net order at the opening quote plus the noise order. Define

$$\Xi_\vartheta := \vartheta\tau L + \xi. \tag{8}$$

By (5) and (1), $\vartheta x(P_I, \delta_0) = \vartheta\tau L$. We call $\Xi_\vartheta$ the order flow at the opening quote, or the order flow for short.

For terminal log-odds $\eta \in \mathbb{R}$, let $\bar{\delta} = \bar{\delta}(\sigma(\eta))$ denote the corresponding execution price in (4). Define

$$\Psi_b(\eta) := b(\eta - \operatorname{logit}\delta_0) + \tau(\operatorname{logit}\bar{\delta} - \operatorname{logit}\delta_0). \tag{9}$$

When the batch clears at this execution price, $\Psi_b(\eta)$ equals the aggregate order the traders would place were every unit executed at $\delta_0$, plus the noise order. This hypothetical demand holds traders' posteriors fixed at their equilibrium values.[10]

**Lemma 4** (The price reveals the order flow at the opening quote)**.** *In any equilibrium, in every state, the order flow at the opening quote is*

$$\Xi_\vartheta \;=\; \Psi_b(\operatorname{logit}\delta^*) - (1-\vartheta)\tau(\operatorname{logit}P_U(\bar{\delta}) - \operatorname{logit}\delta_0), \tag{10}$$

*where $\delta^*$ is the terminal quote in that state.*

*Proof.* Appendix A.5.

The execution price is strictly increasing in the terminal quote (Lemma 1(iii)), so ob-

[10] By Definition 1(iii), the total net order is $x = \int x(P_i, \bar{\delta})\, di + \xi$. The LMSR pricing rule implies $\eta = \operatorname{logit}\delta_0 + x/b$, hence $x = b(\eta - \operatorname{logit}\delta_0)$. By (5), holding $P_i$ fixed gives $x(P_i, \delta_0) - x(P_i, \bar{\delta}) = \tau(\operatorname{logit}\bar{\delta} - \operatorname{logit}\delta_0)$. Integrating over the unit mass of traders yields

$$\int x(P_i, \delta_0)\, di + \xi = x + \tau(\operatorname{logit}\bar{\delta} - \operatorname{logit}\delta_0) = \Psi_b(\eta).$$

serving the execution price identifies the terminal quote. The uninformed posterior $P_U(\bar{\delta})$ is also a function of the execution price. Thus, every term on the right side of (10) can be recovered from the execution price. Subtracting uninformed traders' hypothetical demand at $\delta_0$ from $\Psi_b(\operatorname{logit}\delta^*)$ leaves informed traders' hypothetical demand at $\delta_0$ plus the noise order, $\Xi_\vartheta$. Theorem 1 shows that in the Gaussian benchmark the execution price in a continuous equilibrium reveals exactly $\Xi_\vartheta$.

## 3.1 Existence and uniqueness

In the Gaussian benchmark, the order flow at the opening quote, $\Xi_\vartheta$, is normally distributed conditional on $B$, with mean $(2B-1)\vartheta\tau\bar{\rho}^2/2$. Define

$$v_\vartheta := \vartheta^2\tau^2\bar{\rho}^2 + \sigma_\xi^2, \qquad \beta_\vartheta := \frac{\vartheta\tau\bar{\rho}^2}{v_\vartheta}, \qquad \varpi_\vartheta := 1 + (1-\vartheta)\tau\beta_\vartheta. \tag{11}$$

Here $v_\vartheta$ is the conditional variance of the order flow. Bayes' rule gives

$$\operatorname{logit}\Pr(B=1 \mid \Xi_\vartheta) = \operatorname{logit}\delta_0 + \beta_\vartheta\Xi_\vartheta,$$

so $\beta_\vartheta$ measures the change in posterior log-odds per unit of observed order flow. When uninformed traders condition on $\Xi_\vartheta$, their aggregate demand at the opening quote is $(1-\vartheta)\tau\beta_\vartheta\Xi_\vartheta$. Adding this demand to $\Xi_\vartheta$ gives $\varpi_\vartheta\Xi_\vartheta$, the aggregate demand at the opening quote including the noise order.

**Theorem 1** (Equilibrium at a fixed informed fraction)**.** *Under Assumption 1, for every $\vartheta \in [0,1]$ and $b > 0$, a continuous equilibrium exists. In every continuous equilibrium:*

*(i) Prices are uniquely determined by*

$$\operatorname{logit} \delta^* = \Psi_b^{-1}(\varpi_\vartheta \Xi_\vartheta) \tag{12}$$

*and the execution-price formula* (4)*. If* $\vartheta < 1$*, the uninformed posterior is unique and satisfies* $\operatorname{logit} P_U(\bar{\delta}) = \operatorname{logit} \delta_0 + \beta_\vartheta \Xi_\vartheta$ *in every state.*

*(ii) Both prices are smooth, strictly increasing functions of* $\Xi_\vartheta$*. Observing either price is equivalent to observing* $\Xi_\vartheta$*.*

*(iii) For fixed* $\vartheta$*, the information revealed by either price does not depend on* $b$*.*

*Proof.* Appendix A.6.

In (12), written as $\Psi_b(\operatorname{logit} \delta^*) = \varpi_\vartheta \Xi_\vartheta$, both sides measure hypothetical aggregate demand, including the noise order, if every unit traded at $\delta_0$ while traders retained their equilibrium posteriors. The left side equals the net order actually executed plus the change in aggregate demand when $\delta_0$ replaces $\bar{\delta}$ in traders' demand schedules. Greater liquidity requires a larger net purchase or sale to produce the same movement in the terminal quote.

The right side includes the demand generated by uninformed traders' inference from the price. Each additional unit of $\Xi_\vartheta$ raises uninformed traders' posterior log-odds by $\beta_\vartheta$. At $\delta_0$, each uninformed trader therefore increases her demand by $\tau\beta_\vartheta$. Across the fraction $1-\vartheta$ of uninformed traders, the increase in demand is $(1-\vartheta)\tau\beta_\vartheta$ per unit of order flow. The factor $\varpi_\vartheta$ therefore measures the amplification of $\Xi_\vartheta$ through uninformed traders' inference. When all traders are informed, $\varpi_\vartheta = 1$ and hypothetical aggregate demand equals $\Xi_\vartheta$. A higher noise variance lowers $\beta_\vartheta$, so uninformed traders' demand at $\delta_0$ becomes less sensitive to order flow.

The execution price and the terminal quote reveal exactly $\Xi_\vartheta$. Neither price fully reveals $L$, because order flow also contains the noise order. For fixed $\vartheta$, liquidity changes how far prices move for a given order flow, while the information revealed by either price remains unchanged.[11]

In the Gaussian benchmark with $0 < \vartheta < 1$, an informed trader and an uninformed trader submit net YES orders $x_I$ and $x_U$, respectively. At the execution price $\bar{\delta}$ characterized in Theorem 1, their optimal orders in (5) and the total net order satisfy

$$\begin{aligned} x_I &= \tau(\operatorname{logit}\delta_0 + L - \operatorname{logit}\bar{\delta}), \\ x_U &= \tau(\operatorname{logit}\delta_0 + \beta_\vartheta \Xi_\vartheta - \operatorname{logit}\bar{\delta}), \\ x &= b(\operatorname{logit}\delta^* - \operatorname{logit}\delta_0). \end{aligned} \tag{13}$$

The total net order $x$ includes the noise order, so market clearing requires $\vartheta x_I + (1-\vartheta)x_U + \xi = x$ in every state.

Informed and uninformed traders face the same execution price, so their order difference, $x_I - x_U = \tau(L - \beta_\vartheta \Xi_\vartheta)$ in (13), reflects the information in the signal beyond that revealed by the price.

For the uninformed trader, a higher posterior probability of $B = 1$ raises demand at a given execution price, while a higher execution price lowers demand at a given posterior. Her posterior log-odds change at rate $\beta_\vartheta$ with order flow. When the execution price's log-odds increase at a rate greater than $\beta_\vartheta$, her net YES order decreases in response to a marginal

[11] The price function is unique within the class of continuous equilibria. As Breon-Drish (2015) notes, the standard noisy rational expectations model admits discontinuous equilibria; our argument does not exclude them. Theorem 7 in Appendix A.7 extends the result beyond the Gaussian benchmark. Under its assumptions, a continuous equilibrium exists if and only if $G_\vartheta$ in (30) is one-to-one. Here $G_\vartheta(u)$ is total hypothetical demand at $\delta_0$, including the noise order, when $\Xi_\vartheta = u$ and uninformed traders condition their beliefs on $u$. A log-concave noise density, whose logarithm is concave, ensures that $G_\vartheta$ is one-to-one; the normal, logistic, and Laplace densities are examples. Whenever a continuous equilibrium exists under Theorem 7, its price function is unique and both prices reveal exactly $\Xi_\vartheta$ for every $b > 0$.

increase in order flow.[12]

The equilibrium characterization also determines the distributions of prices and the total net order. Let $\Phi$ and $\phi$ denote the standard normal distribution function and density. For terminal log-odds $\eta \in \mathbb{R}$ and event outcome $B \in \{0, 1\}$, define

$$z_B(\eta) := \frac{\Psi_b(\eta)/\varpi_\vartheta - (2B-1)\vartheta\tau\bar{\rho}^2/2}{\sqrt{v_\vartheta}}. \tag{14}$$

For $p \in (0, 1)$, let $\bar{\eta}(p)$ be the terminal log-odds corresponding to execution price $p$, so that $\bar{\delta}(\sigma(\bar{\eta}(p))) = p$. For $m \in \mathbb{R}$, write $\eta_m := \operatorname{logit} \delta_0 + m/b$.

**Proposition 1** (Distributions of prices and the total net order)**.** *Under Assumption 1, fix $\vartheta \in [0, 1]$ and $b > 0$. In any continuous equilibrium characterized by Theorem 1:*

*(i) Conditional on $B$, the terminal quote, execution price, and total net order have densities*

$$\begin{aligned} f_{\delta^*|B}(d) &= \phi(z_B(\operatorname{logit} d))\frac{\Psi_b'(\operatorname{logit} d)}{\varpi_\vartheta\sqrt{v_\vartheta}\, d(1-d)}, \\ f_{\bar{\delta}|B}(p) &= \phi(z_B(\bar{\eta}(p)))\frac{\Psi_b'(\bar{\eta}(p))}{\varpi_\vartheta\sqrt{v_\vartheta}}\bar{\eta}'(p), \\ f_{x|B}(m) &= \phi(z_B(\eta_m))\frac{\Psi_b'(\eta_m)}{\varpi_\vartheta\sqrt{v_\vartheta}\, b}, \end{aligned}$$

*for $d, p \in (0, 1)$ and $m \in \mathbb{R}$.*

*(ii) For each of the three variables, the unconditional density is the mixture of its conditional densities given $B = 1$ and $B = 0$, with weights $\delta_0$ and $1 - \delta_0$, respectively.*

*Proof.* Appendix A.8.

[12] Lemma 6 in Appendix A.9 gives the distributions of individual orders.

## 3.2 The value of information

Fix $\vartheta \in [0,1]$ and an equilibrium of Definition 2. By Lemma 2, informed and uninformed traders' conditional certainty-equivalent gains from trading are $\tau D_I$ and $\tau D_U$, respectively, where

$$D_I := \mathrm{KL}(\bar{\delta} \,\|\, P_I), \qquad D_U := \mathrm{KL}(\bar{\delta} \,\|\, P_U(\bar{\delta})). \tag{15}$$

Define the corresponding ex ante certainty-equivalent gains and the value of the signal by

$$G_I := -\tau \log \mathbb{E}[e^{-D_I}], \qquad G_U := -\tau \log \mathbb{E}[e^{-D_U}], \qquad \Delta_G := G_I - G_U, \tag{16}$$

where expectations are taken before information acquisition.

**Proposition 2** (The value of the signal)**.** *Fix $\vartheta \in [0,1]$ and an equilibrium of Definition 2.*

*(i) The ex ante certainty equivalent of terminal wealth is $W - c_I + G_I$ if a trader acquires the signal and $W + G_U$ otherwise.*

*(ii) $\Delta_G \geq 0$, with equality if and only if $P_I = P_U(\bar{\delta})$ almost surely.*

*Proof.* Appendix A.10.

A trader is willing to acquire the signal if and only if $\Delta_G \geq c_I$. An informed trader can follow an uninformed trader's demand schedule, so the signal has nonnegative value before acquisition costs. Because traders are risk averse, the signal's value is a difference of ex ante certainty equivalents, which generally differs from the expected difference between conditional certainty equivalents, $\tau\mathbb{E}[D_I - D_U]$.

In the Gaussian benchmark, observing the execution price is equivalent to observing $\Xi_\vartheta$ (Theorem 1). An uninformed trader therefore revises her prior log-odds by $\beta_\vartheta \Xi_\vartheta$. The residual $L - \beta_\vartheta \Xi_\vartheta$ is the additional change in log-odds available to an informed trader after

observing the price. Residual precision measures the information the signal provides beyond the price.

The two components have the Gaussian form in Assumption 1, with respective public and residual precisions (Lemma 6(i))

$$\bar{\rho}_{U,\vartheta} := \frac{\vartheta\tau\bar{\rho}^2}{\sqrt{v_\vartheta}}, \qquad \bar{\rho}_{I,\vartheta} := \frac{\bar{\rho}\,\sigma_\xi}{\sqrt{v_\vartheta}}, \qquad \bar{\rho}^2_{U,\vartheta} + \bar{\rho}^2_{I,\vartheta} = \bar{\rho}^2. \tag{17}$$

The next proposition compares the information revealed by the price across informed fractions and noise-trading volatilities.[13]

**Proposition 3.** *Under Assumption 1, in the equilibrium of Theorem 1, for every $\vartheta \in [0,1]$ and $b > 0$, the price is as informative about $B$ as a Gaussian signal with precision $\bar{\rho}_{U,\vartheta}$ in (17). For $\vartheta \in (0,1]$,*

$$\partial_\vartheta \bar{\rho}_{U,\vartheta} > 0, \qquad \partial_{\sigma_\xi} \bar{\rho}_{U,\vartheta} < 0, \qquad \partial_b \bar{\rho}_{U,\vartheta} = 0.$$

*Every measure of price informativeness that respects garbling, including mutual information between the price and $B$, is weakly increasing in $\vartheta$ and weakly decreasing in $\sigma_\xi$.*

*Proof.* Appendix A.11.

A larger informed fraction leaves less of the signal private, whereas greater noise-trading volatility preserves more of its private information.

**Theorem 2** (The value of information in the Gaussian benchmark)**.** *Under Assumption 1, in the equilibrium of Theorem 1, for every $\vartheta \in [0,1]$ and $b > 0$:*

[13] A signal $Y$ is a *garbling* of a signal $X$ if $Y$ can be generated from $X$ using a conditional distribution that does not depend on $B$. A measure of informativeness respects garbling if it assigns $Y$ no greater informativeness than $X$.

*(i) The value of the signal is*

$$\Delta_G = -\tau \log \frac{\mathbb{E}[e^{-D_U} \exp(-\frac{1}{2}\bar{\rho}^2_{I,\vartheta}\bar{\delta}(1-\bar{\delta}))]}{\mathbb{E}[e^{-D_U}]}. \tag{18}$$

*(ii)* $0 < \Delta_G < \tau\bar{\rho}^2_{I,\vartheta}/8$.

*(iii) As* $b \to \infty$,

$$\Delta_G \longrightarrow \frac{\tau\delta_0(1-\delta_0)\bar{\rho}^2_{I,\vartheta}}{2}$$

*uniformly in* $\vartheta \in [0,1]$.

*Proof.* Appendix A.12.

The execution price determines the stake $\bar{\delta}(1-\bar{\delta})$, the variance of the contract's payoff under a belief equal to the price. For a given residual precision, the certainty-equivalent gain from information conditional on the price is greatest at $\bar{\delta} = 1/2$ and approaches zero as the price approaches either endpoint.

Risk aversion enters through the weights $e^{-D_U}$, which are proportional to the uninformed trader's expected marginal utility conditional on the price. States in which an uninformed trader earns a larger certainty-equivalent gain $\tau D_U$ receive less weight in the ex ante valuation.

Greater liquidity brings the execution price closer to the opening quote without changing the information revealed by the price at a fixed informed fraction (Proposition 4(i) and Theorem 1(iii)). As $b \to \infty$, the stake converges to $\delta_0(1-\delta_0)$ in every state. Differences in marginal utility across states therefore no longer affect the limiting value of information in part (iii).

## 3.3 Comparative statics

We examine how liquidity and noise-trading volatility affect prices, trading, and the value of information at a fixed informed fraction.

Throughout, we use the equilibrium characterized in Theorem 1. In each state, the order flow $\Xi_\vartheta$ in (8) determines the log-odds displacements of the terminal quote, $\operatorname{logit}\delta^* - \operatorname{logit}\delta_0$, and the execution price, $\operatorname{logit}\bar{\delta} - \operatorname{logit}\delta_0$. Both displacements vanish exactly when $\Xi_\vartheta = 0$. The *secant ratio* measures the execution price's log-odds displacement relative to that of the terminal quote:[14]

$$\varsigma(\delta^*) \;:=\; \frac{\operatorname{logit}\bar{\delta} - \operatorname{logit}\delta_0}{\operatorname{logit}\delta^* - \operatorname{logit}\delta_0} \quad (\delta^* \neq \delta_0), \qquad \varsigma(\delta_0) \;:=\; \frac{1}{2}, \qquad \bar{\varsigma}(\delta_0) \;:=\; \sup_{\delta^* \in (0,1)} \varsigma(\delta^*). \tag{19}$$

### 3.3.1 Liquidity

We first consider how liquidity affects prices and the total net order.

**Proposition 4.** *Under Assumption 1, fix $\vartheta \in [0,1]$ and a state with $\Xi_\vartheta = u \neq 0$.*

*(i) The log-odds displacements $\operatorname{logit}\delta^* - \operatorname{logit}\delta_0$ and $\operatorname{logit}\bar{\delta} - \operatorname{logit}\delta_0$ have the sign of $u$. Their absolute values and $|\bar{\delta} - \delta_0|$ are strictly decreasing in $b$.*

*(ii) The magnitude of the total net order, $|x| = b\,|\operatorname{logit}\delta^* - \operatorname{logit}\delta_0|$, is strictly increasing in $b$, with limits $0$ as $b \to 0$ and $\varpi_\vartheta|u|$ as $b \to \infty$.*

*Proof.* Appendix A.15.

Greater liquidity allows the market maker to accommodate a larger net purchase or sale with a smaller departure of prices from the opening quote. As liquidity vanishes, traders

[14] At the prior, the ratio equals the derivative of $\operatorname{logit}\bar{\delta}$ with respect to $\operatorname{logit}\delta^*$; see Lemma 8 in Appendix A.14.

absorb the noise order among themselves, with vanishing net trade with the market maker. As liquidity grows without bound, the market maker meets aggregate demand, including the noise order, at a price approaching the opening quote.

The smaller price movements induced by greater liquidity do not reduce the information available to uninformed traders. For every $b > 0$, either price reveals the same order flow $\Xi_\vartheta$ (Theorem 1(ii)–(iii)).

The next proposition characterizes when an uninformed trader trades in the direction of order flow and when she trades against it.

**Proposition 5.** *Under Assumption 1, fix $\vartheta \in (0, 1]$ and $b > 0$.*

*(i) In a state with $\Xi_\vartheta = u \neq 0$ and terminal quote $\delta^*$, the uninformed order $x_U$ has the sign of $u$ if $\varsigma(\delta^*) < \vartheta\tau\bar{\rho}^2 b/\sigma_\xi^2$, the opposite sign if the inequality is reversed, and is zero at equality.*

*(ii) If $\sigma_\xi^2 < \vartheta\tau\bar{\rho}^2 b/\bar{\varsigma}(\delta_0)$, $x_U$ has the sign of $\Xi_\vartheta$ in every state with $\Xi_\vartheta \neq 0$. If $\sigma_\xi^2 > \vartheta\tau\bar{\rho}^2 b/\bar{\varsigma}(\delta_0)$, $x_U$ has the opposite sign in some state.*

*(iii) There is no pair $(b, \sigma_\xi)$ at which $x_U$ is zero in every state. At fixed $\sigma_\xi$, $\Pr(x_U \Xi_\vartheta < 0) \to 1$ as $b \to 0$.*

*Proof.* Appendix A.16.

An uninformed trader buys when her posterior exceeds the execution price and sells when her posterior is lower. She trades in the direction of order flow when her posterior is farther from the opening quote than the execution price. When the execution price overshoots her posterior, she trades against order flow and supplies liquidity.

Greater liquidity brings the execution price closer to the opening quote without changing the uninformed posterior. Part (ii) identifies when liquidity is sufficiently large relative to

noise variance for uninformed traders to follow every nonzero order flow.

We next examine how liquidity affects the value of the signal at a fixed informed fraction. Write $a_U := e^{-D_U}$ and $a_I := a_U \exp(-\frac{1}{2}\bar{\rho}^2_{I,\vartheta}\bar{\delta}(1-\bar{\delta}))$. Differentiating (18) with respect to $b$ gives[15]

$$\partial_b \Delta_G \;=\; \frac{\tau\bar{\rho}^2_{I,\vartheta}}{2}\,\frac{\mathbb{E}[a_I\,\partial_b(\bar{\delta}(1-\bar{\delta}))]}{\mathbb{E}[a_I]} \;+\; \tau\Big(\frac{\mathbb{E}[a_I\,\partial_b D_U]}{\mathbb{E}[a_I]} - \frac{\mathbb{E}[a_U\,\partial_b D_U]}{\mathbb{E}[a_U]}\Big). \tag{20}$$

The first term reflects changes in the stake $\bar{\delta}(1-\bar{\delta})$. Greater liquidity moves the execution price toward the opening quote. The stake rises when this movement brings the execution price closer to one-half and falls when it takes the execution price farther from one-half.

The second term reflects changes in the weights used to value the signal before acquisition. Liquidity changes the uninformed trader's certainty-equivalent gain $\tau D_U$ and hence the weight $a_U = e^{-D_U}$; states with larger uninformed gains receive less weight.

As liquidity grows without bound, the execution price approaches the prior and the stake becomes the same in every state. Changes in the relative weights assigned to states then have no effect on the value of the signal.[16]

### 3.3.2 Noise-trading volatility

Noise-trading volatility affects both the distribution of order flow and the information that uninformed traders infer from it. The next proposition distinguishes the price response to a given order flow from the behavior of prices as noise grows without bound.

**Proposition 6.** *Under Assumption 1, fix $b > 0$.*

[15]The derivatives inside the expectations hold $\Xi_\vartheta$ fixed. Its distribution and the residual precision do not depend on $b$; Lemma 7 justifies differentiation under the expectation.

[16]In this limit, $a_I/a_U = \exp(-\frac{1}{2}\bar{\rho}^2_{I,\vartheta}\delta_0(1-\delta_0))$ is constant across states, so $a_I/\mathbb{E}[a_I] = a_U/\mathbb{E}[a_U]$. The weight terms therefore cancel in the first derivative with respect to inverse liquidity $1/b$ at $1/b = 0$. At the symmetric prior, this derivative is zero and the weight terms also cancel in the second-order expansion; see Appendix A.20.

*(i) For fixed $\vartheta \in (0,1)$ and order flow $\Xi_\vartheta = u \neq 0$, the log-odds displacements* $\operatorname{logit}\delta^* - \operatorname{logit}\delta_0$ *and* $\operatorname{logit}\bar{\delta} - \operatorname{logit}\delta_0$ *are strictly decreasing in absolute value as $\sigma_\xi$ rises.*

*(ii) Fix $\vartheta \in [0,1]$. As $\sigma_\xi \to \infty$, for each $k \in \{0,1\}$,*

$$\bar{\delta} - \mathbf{1}\{\Xi_\vartheta > 0\} \longrightarrow 0 \quad \textit{in probability conditional on } B = k,$$
$$\Pr(\Xi_\vartheta > 0 \mid B = k) \longrightarrow \frac{1}{2}.$$

*Moreover, $\bar{\rho}_{U,\vartheta} \to 0$.*

*Proof.* Appendix A.17.

For a given order flow, greater noise-trading volatility brings the uninformed posterior closer to the prior. Uninformed traders' demand evaluated at the opening quote therefore falls in absolute value, reducing $\varpi_\vartheta$. Both the execution price and the terminal quote consequently remain closer to the opening quote.

Greater noise-trading volatility also increases the conditional variance of order flow. At high noise, large positive order flows push the execution price toward one, and large negative order flows push it toward zero. Prices near either endpoint become equally likely under both event outcomes, while the posterior inferred from the price converges to the prior.

In the CARA-normal model of Grossman and Stiglitz (1980), more noise raises the value of information at a fixed informed fraction by reducing what uninformed traders learn from prices. The value depends on conditional variances, with no separate dependence on the realized price. Here, noise likewise increases residual precision, but the binary payoff makes the value of the residual signal depend on the stake $\bar{\delta}(1-\bar{\delta})$. At any finite LMSR liquidity, sufficiently large noise orders move the execution price toward zero or one, where the stake vanishes. As Theorem 3 will show, the vanishing stake drives the value of information to zero despite the increase in residual precision.

We next characterize the value of the signal as noise-trading volatility varies.

**Theorem 3.** *Under Assumption 1, fix $\vartheta \in (0,1]$ and $b > 0$, and regard $\Delta_G$ as a function of $\sigma_\xi$.*

*(i) $\Delta_G \to 0$ as $\sigma_\xi \to 0$ and as $\sigma_\xi \to \infty$.*

*(ii) $\Delta_G$ attains a global maximum at some $\sigma_\xi \in (0,\infty)$, and $\partial_{\sigma_\xi}\Delta_G < 0$ at some $\sigma_\xi > 0$.*

*Proof.* Appendix A.18.

As $\sigma_\xi \to 0$, the price reveals progressively more of the informed traders' signal, and the gain from acquiring it tends to zero. As $\sigma_\xi \to \infty$, residual precision $\bar{\rho}_{I,\vartheta}$ approaches $\bar{\rho}$, but the stake $\bar{\delta}(1-\bar{\delta})$ converges to zero in probability at every fixed $b > 0$ (Proposition 6(ii)). The signal remains informative about the event, but its incremental trading value tends to zero.

The order of limits matters. Taking $b \to \infty$ first fixes the limiting stake at $\delta_0(1-\delta_0)$ (Theorem 2(iii)). Letting $\sigma_\xi \to \infty$ afterward gives a signal value of $\tau\delta_0(1-\delta_0)\bar{\rho}^2/2 > 0$, whereas letting $\sigma_\xi \to \infty$ at any fixed $b > 0$ renders the signal value zero.

# 4. Information acquisition

We now endogenize the informed fraction. Each trader decides whether to acquire the signal before trading, taking the informed fraction and the execution price in each state as given. Write $\Delta_G(\vartheta, b)$ for the value of the signal defined in (16), evaluated in the equilibrium of Theorem 1 at informed fraction $\vartheta$ and liquidity $b$. Under Assumption 1, $\Delta_G$ is a smooth function of $(\vartheta, b, \sigma_\xi, \bar{\rho})$ on $[0,1] \times (0,\infty)^3$ (Lemma 7 in Appendix A.13).

**Definition 3** (Acquisition equilibrium)**.** *Fix $b > 0$. An* acquisition equilibrium *consists of a fraction $\vartheta^* \in [0,1]$ and a continuous equilibrium of Theorem 1 at $\vartheta^*$, satisfying*

$$\begin{cases} \Delta_G(0,b) \leq c_I, & \text{if } \vartheta^* = 0, \\ \Delta_G(\vartheta^*,b) = c_I, & \text{if } \vartheta^* \in (0,1), \\ \Delta_G(1,b) \geq c_I, & \text{if } \vartheta^* = 1. \end{cases}$$

As in Grossman and Stiglitz (1980), no trader can gain by changing her acquisition decision. At an interior equilibrium, each trader is indifferent ex ante between acquiring the signal and remaining uninformed.

## 4.1 Existence

Proposition 7 establishes that an acquisition equilibrium exists at every liquidity and that, where the value of the signal has a nonzero slope in the informed fraction, an interior equilibrium varies smoothly with the parameters.

**Proposition 7** (Information acquisition)**.** *Under Assumption 1:*

*(i) For every $b > 0$, an acquisition equilibrium exists. If $\Delta_G(0,b) > c_I > \Delta_G(1,b)$, an interior acquisition equilibrium exists.*

*(ii) At an interior acquisition equilibrium with $\partial_\vartheta \Delta_G(\vartheta^*,b) \neq 0$, the informed fraction $\vartheta^*$ is locally a smooth function of $(b, \sigma_\xi, \bar{\rho}, c_I)$, with*

$$\begin{aligned} \frac{d\vartheta^*}{dz} &= -\frac{\partial_z \Delta_G}{\partial_\vartheta \Delta_G}, \quad z \in \{b, \sigma_\xi, \bar{\rho}\}, \\ \frac{d\vartheta^*}{dc_I} &= \frac{1}{\partial_\vartheta \Delta_G}. \end{aligned} \tag{21}$$

*All partial derivatives are evaluated at the acquisition equilibrium.*

*(iii) If $c_I \geq \tau\bar{\rho}^2/8$, then $\vartheta^* = 0$ is the unique acquisition equilibrium for every $b > 0$ and $\sigma_\xi > 0$.*

*Proof.* Appendix A.19.

At an interior equilibrium with $\partial_\vartheta \Delta_G < 0$, a small change in liquidity, noise, or signal precision increases the informed fraction if it raises the value of the signal at a fixed informed fraction. Additional acquisition offsets this increase in value, preserving traders' indifference between acquiring the signal and remaining uninformed. A higher acquisition cost instead reduces the informed fraction. For costs at or above the threshold in part (iii), neither greater liquidity nor more noise can induce traders to acquire the signal.

## 4.2 Equilibrium uniqueness and acquisition in deep and low-noise markets

We first show that a sufficiently deep market has at most one acquisition equilibrium with a fraction of informed traders above a fixed positive lower bound.

**Theorem 4.** *Under Assumption 1, for every $\underline{\vartheta} \in (0, 1]$, there is $b_{\underline{\vartheta}} < \infty$ such that, for every $b > b_{\underline{\vartheta}}$:*

*(i) For all $\vartheta \in [\underline{\vartheta}, 1]$,*

$$\partial_b \Delta_G(\vartheta, b) > 0 \qquad \text{and} \qquad \partial_\vartheta \Delta_G(\vartheta, b) < 0.$$

*If $\delta_0 = 1/2$, the first inequality holds for all $\vartheta \in [0, 1]$.*

*(ii) The equation $\Delta_G(\vartheta, b) = c_I$ has at most one solution in $[\underline{\vartheta}, 1]$. At any interior acquisition equilibrium $\vartheta^* \in [\underline{\vartheta}, 1)$,*

$$\frac{d\vartheta^*}{db} > 0 \qquad \textit{and} \qquad \frac{d\vartheta^*}{dc_I} < 0.$$

*Proof.* Appendix A.20.

Greater liquidity keeps the execution price closer to the opening quote. When the prior is asymmetric, informed demand at the opening quote favors the more likely outcome on average. In a deep market, moderating price movements toward that outcome raises the average stake $\bar{\delta}(1-\bar{\delta})$. At the symmetric prior, every departure from one-half lowers the stake. In a deep market, greater liquidity therefore raises the value of the residual signal in either case.

At an interior acquisition equilibrium, the increase in the value of information induces more traders to acquire the signal. Their orders make the price more informative and reduce the private benefit of acquisition. Liquidity affects price informativeness through this acquisition response: at a fixed informed fraction, the price reveals the same information at every liquidity (Theorem 1(iii)).

We next establish conditions for a unique acquisition equilibrium when noise trading is small. The signal-to-noise ratio in order flow $\Xi_\vartheta = \vartheta\tau L + \xi$ compares the conditional standard deviations of its informed and noise components:

$$\varkappa_\vartheta := \frac{\vartheta\tau\bar{\rho}}{\sigma_\xi} = \frac{\bar{\rho}_{U,\vartheta}}{\bar{\rho}_{I,\vartheta}}. \tag{22}$$

For fixed $b > 0$ and $\varkappa \geq 0$, define the limiting value of the signal by

$$\mathscr{V}_b(\varkappa) := \lim_{\sigma_\xi \downarrow 0} \Delta_G \left( \frac{\varkappa \sigma_\xi}{\tau \bar{\rho}}, b \right),$$

where $\Delta_G$ is evaluated at noise-trading volatility $\sigma_\xi$.[17] The public and residual precisions remain fixed as noise trading and the informed fraction vanish. At $\varkappa = 0$, the limiting execution price equals the opening quote, so $\mathscr{V}_b(0) = \tau \delta_0 (1 - \delta_0) \bar{\rho}^2 / 2$.

**Theorem 5.** *Under Assumption 1, fix $\delta_0$, $\tau$, $\bar{\rho}$, and $c_I$, with $0 < c_I < \mathscr{V}_b(0)$.*

*(i) For every fixed $b > 0$, all acquisition equilibria are interior for sufficiently small $\sigma_\xi$, and their informed fractions converge to zero as $\sigma_\xi \to 0$.*

*(ii) There is $\bar{b} < \infty$ such that, for every $b > \bar{b}$, the equation $\mathscr{V}_b(\varkappa) = c_I$ has a unique solution $\varkappa_b > 0$. For each such $b$, the acquisition equilibrium is unique and interior for sufficiently small $\sigma_\xi$, with*

$$\frac{\vartheta^*}{\sigma_\xi} \longrightarrow \frac{\varkappa_b}{\tau \bar{\rho}} \quad \text{as } \sigma_\xi \to 0, \qquad \frac{d\vartheta^*}{d\sigma_\xi} > 0.$$

*(iii) As $b \to \infty$,*

$$\varkappa_b \longrightarrow \varkappa_\infty := \sqrt{\frac{\mathscr{V}_b(0)}{c_I} - 1}.$$

*Proof.* Appendix A.21.

At a fixed positive informed fraction, vanishing noise would make the price fully revealing and eliminate the private value of the signal. Information acquisition must therefore also vanish if traders are to recover the signal's cost.

[17] Appendix A.21 establishes this limit and gives an integral expression for $\mathscr{V}_b$.

For a fixed liquidity $b > \bar{b}$, the price remains partially informative as $\sigma_\xi \to 0$: fewer traders acquire the signal, but their orders also face less noise. The residual signal retains a private value equal to the acquisition cost.

At a fixed noise-trading volatility, the next corollary establishes uniqueness for sufficiently large liquidity and identifies the limiting informed fraction.

**Corollary 1.** *Under Assumption 1, fix $\delta_0, \tau, \bar{\rho}, \sigma_\xi, c_I$ with $c_I < \mathcal{V}_b(0)$ and $\vartheta_\infty := \varkappa_\infty \sigma_\xi/(\tau\bar{\rho}) < 1$. There is $b' < \infty$ such that the acquisition equilibrium is unique and interior for $b > b'$, with $\vartheta^*(b) \to \vartheta_\infty$ as $b \to \infty$.*

*The limiting precisions satisfy*

$$\bar{\rho}^2_{I,\vartheta_\infty} = \frac{2c_I}{\tau\delta_0(1-\delta_0)}, \qquad \bar{\rho}^2_{U,\vartheta_\infty} = \bar{\rho}^2 - \frac{2c_I}{\tau\delta_0(1-\delta_0)}.$$

*Proof.* Appendix A.22.

The execution price approaches the opening quote as liquidity grows. In the limit, traders' indifference about acquisition fixes the residual precision needed to cover the signal's cost. A higher acquisition cost requires more information to remain private and leaves the price less informative.

The next result characterizes the response of information acquisition to noise in a sufficiently deep market.

**Theorem 6.** *Under Assumption 1, fix $\delta_0, \tau, \bar{\rho}$, and $c_I$. For every compact $K \subset (0,1] \times (0,\infty)$ of pairs $(\vartheta, \sigma_\xi)$, there is $b_K < \infty$ such that, for $b > b_K$ and $(\vartheta, \sigma_\xi) \in K$,*

$$\partial_{\sigma_\xi}\Delta_G > 0 \qquad \text{and} \qquad \partial_\vartheta \Delta_G < 0.$$

*At every interior acquisition equilibrium with* $(\vartheta^*, \sigma_\xi) \in K$ *and* $b > b_K$*,*

$$\frac{d\vartheta^*}{d\sigma_\xi} > 0.$$

*Proof.* Appendix A.23.

At an interior acquisition equilibrium in a sufficiently deep market, greater noise-trading volatility induces more traders to acquire the signal, as in Grossman and Stiglitz (1980). In the infinite-liquidity limit of Corollary 1, the informed fraction increases in proportion to noise-trading volatility. This additional acquisition exactly offsets the loss of price informativeness caused by noise, leaving public precision unchanged as long as the limiting acquisition equilibrium remains interior.

# 5. Market maker profits and price accuracy

We next study the market maker's trading profits and the accuracy of the execution price as a forecast. The market maker faces the total net order $x = \vartheta x_I + (1-\vartheta)x_U + \xi$, including informed, uninformed, and noise trades. Its loss after settlement is

$$\mathscr{L} := xB - \Delta C = x(B - \bar{\delta}), \tag{23}$$

where $\Delta C = x\bar{\delta}$ is its net trading revenue (Lemma 1(ii)). When $x > 0$, the market maker sells $x$ shares at $\bar{\delta}$ and pays $B$ per share at settlement; when $x < 0$, it buys $-x$ shares at $\bar{\delta}$ and receives their settlement value. A negative value of $\mathscr{L}$ is a trading profit.

**Proposition 8.** *Under Assumption 1, in the equilibrium of Theorem 1, fix* $\vartheta \in [0,1]$ *and* $b > 0$*.*

*(i) Conditional on order flow,*

$$\mathbb{E}[\mathscr{L} \mid \Xi_\vartheta] = x(P_U(\bar{\delta}) - \bar{\delta}),$$

*which has the sign of* $x\, x_U$*.*

*(ii) The expected loss satisfies*

$$\begin{aligned}\mathbb{E}[\mathscr{L}] = \vartheta\, \mathbb{E}[x_I(B - \bar{\delta})] + (1 - \vartheta)\, \mathbb{E}[x_U(B - \bar{\delta})] \\ + \mathbb{E}[\xi(B - \bar{\delta})],\end{aligned}$$

*where*

$$\mathbb{E}[x_I(B - \bar{\delta})] > 0, \qquad \mathbb{E}[x_U(B - \bar{\delta})] > 0,$$

*and*

$$\mathbb{E}[\xi(B - \bar{\delta})] = -\sigma_\xi^2\, \mathbb{E}\left[\frac{d\bar{\delta}}{d\Xi_\vartheta}\right] < 0.$$

*(iii) The supremum of* $\mathscr{L}$ *over all states is* $b\, M(\delta_0)$*, where*

$$M(\delta_0) = \max\{-\log \delta_0, -\log(1 - \delta_0)\}.$$

*Proof.* Appendix A.24.

Conditional on order flow, the public posterior $P_U(\bar{\delta})$ is the expected payoff per share. The market maker therefore expects a loss if it is a net seller at an execution price below this posterior or a net buyer at a price above it. An uninformed trader buys when the execution price is below the public posterior and sells when it is above. When the total net order has the same sign as the uninformed order, the market maker sells below the expected payoff

or buys above it, and therefore expects a loss. When the two orders have opposite signs, the market maker buys below the expected payoff or sells above it, and therefore expects a profit. Because $x$ has the sign of $\Xi_\vartheta$, Proposition 5(i) also characterizes the sign of this conditional expected loss.

The market maker's expected loss equals the aggregate expected trading profits of all traders, including noise traders. Informed and uninformed traders earn positive expected trading profits because they buy when the execution price is below their respective posterior valuations and sell when it is above them. These profits exclude information acquisition costs. Noise traders submit orders unrelated to the event, but their purchases raise the price they pay and their sales lower the price they receive. In expectation, the market maker gains from noise traders and loses to informed and uninformed traders.

We next examine how liquidity affects the market maker's expected loss.

**Proposition 9.** *Under Assumption 1, consider the equilibrium of Theorem 1 and write* $w_0 := \delta_0(1-\delta_0)$. *In parts (i)–(ii), fix* $\vartheta \in (0,1]$.

*(i)* Small liquidity. *As* $b \to 0$,

$$\mathbb{E}[\mathscr{L}] = -c_1 b + o(b), \qquad c_1 := \mathbb{E}[\tilde{x}^0(\bar{\delta}^0 - P_U(\bar{\delta}))] > 0,$$

*where*

$$\bar{\delta}^0 := \sigma(\operatorname{logit}\delta_0 + \varpi_\vartheta \Xi_\vartheta/\tau), \qquad \tilde{x}^0 := r^{-1}(\varpi_\vartheta \Xi_\vartheta/\tau),$$

*and* $r$ *is defined in* (36).

*(ii)* Large liquidity. *As* $b \to \infty$,

$$\mathbb{E}[\mathscr{L}] \longrightarrow \varpi_\vartheta \vartheta \tau \bar{\rho}^2 w_0 > 0,$$

*and* $\partial_b \mathbb{E}[\mathscr{L}] > 0$ *for all sufficiently large* $b$.

*(iii)* Endogenous acquisition. *Under the hypotheses of Corollary 1, along its acquisition equilibrium* $\vartheta^*(b)$,

$$\lim_{b\to\infty} \mathbb{E}[\mathscr{L}] = \tau w_0 \bar{\rho}^2 - 2c_I(1-\vartheta_\infty).$$

*Proof.* Appendix A.24.

As liquidity vanishes, the market maker's net position tends to zero, so informed and uninformed traders must together absorb the noise order. The execution price adjusts to clear this demand, whereas the public posterior discounts the information in order flow because of noise. Consequently, the limiting execution price responds more strongly to order flow than the public posterior does:[18] it exceeds the posterior for positive order flow and falls below it for negative order flow. These price differences favor the market maker, which sells in the first case and buys in the second, yielding a positive expected profit at sufficiently small liquidity.

As liquidity grows without bound, the execution price approaches the opening quote. The noise trader's expected loss vanishes, so the market maker's limiting expected loss equals the aggregate expected trading profits of informed and uninformed traders. Greater liquidity raises the market maker's expected loss in a sufficiently deep market.

Along the acquisition equilibrium in part (iii), an uninformed trader's limiting expected trading profit is $\tau w_0 \bar{\rho}^2 - 2c_I$. An informed trader earns an additional $2c_I$ before paying for the signal. The market maker's limiting expected loss is the sum of these per-trader expected trading profits weighted by the respective fractions of informed and uninformed traders.

Consider a user who does not trade but reads the displayed execution price as the probability of the event. For a forecast $p \in (0,1)$ that is a function of the state, define its expected

[18] The limiting log-odds difference is $\operatorname{logit} \bar{\delta}^0 - \operatorname{logit} P_U(\bar{\delta}) = \sigma_\xi^2 \Xi_\vartheta / (\tau v_\vartheta)$.

logarithmic score as

$$\mathcal{S}(p) := \mathbb{E}[B \log p + (1 - B) \log(1 - p)]. \tag{24}$$

A higher expected score indicates greater forecast accuracy. The next proposition compares the execution price with the public posterior and characterizes how liquidity affects forecast accuracy.

**Proposition 10** (The accuracy of the execution price)**.** *Under Assumption 1, consider the equilibrium of Theorem 1 at a fixed informed fraction $\vartheta \in [0, 1]$.*

*(i) For every $b > 0$,*

$$\mathcal{S}(\bar{\delta}) = \mathcal{S}(P_U(\bar{\delta})) - \mathbb{E}[\mathrm{KL}(P_U(\bar{\delta}) \,\|\, \bar{\delta})].$$

*(ii) No $b > 0$ makes $\bar{\delta} = P_U(\bar{\delta})$ in every state.*

*(iii) If $\vartheta > 0$, $\mathcal{S}(\bar{\delta})$ is strictly increasing in $b$ for all sufficiently small $b > 0$ and strictly decreasing for all sufficiently large $b$, with $\mathcal{S}(\bar{\delta}) \to \mathcal{S}(\delta_0)$ as $b \to \infty$. It attains its maximum over $b \in (0, \infty)$ at an interior liquidity.*

*Proof.* Appendix A.25.

At a fixed informed fraction, the price reveals the same information at every liquidity level (Theorem 1(iii)). A user who infers the public posterior from the price therefore obtains the same forecast regardless of liquidity. Using the displayed price directly as the event probability gives a less accurate forecast, whose accuracy depends on liquidity.

With a positive informed fraction, liquidity improves forecast accuracy near zero but reduces it in a sufficiently deep market. In the low-liquidity limit, the execution price lies farther from the prior than the public posterior does. Increasing liquidity initially moderates the price response to order flow and improves its expected score. At sufficiently large liquidity, further increases bring the execution price closer to the prior and reduce its accuracy.

# 6. The market maker's liquidity choice

We now let the market maker choose liquidity before information acquisition and trading. The market maker values forecast accuracy for users who do not trade and use the displayed execution price as the event probability. Its accuracy gain is therefore $\mathcal{S}(\bar{\delta}) - \mathcal{S}(\delta_0)$, the improvement in the expected logarithmic score over the prior.[19]

The market maker also bears the expected trading loss $\mathbb{E}[\mathscr{L}]$ and holds the reserve $bM(\delta_0)$ required to cover its loss in every state (Proposition 8(iii)). Carrying this reserve until settlement costs $\gamma > 0$ per unit of capital, giving a capital charge $c_0 b$, where $c_0 := \gamma M(\delta_0)$. The market maker chooses liquidity to solve

$$\max_{b \geq 0} V(b), \qquad V(b) := \lambda[\mathcal{S}(\bar{\delta}) - \mathcal{S}(\delta_0)] - \mathbb{E}[\mathscr{L}] - c_0 b \quad \text{for } b > 0, \qquad V(0) := 0. \tag{25}$$

Here $\lambda > 0$ is the market maker's value per unit of accuracy gain.[20] The score and expected loss are evaluated in a rational expectations equilibrium with endogenous information acquisition, where $\vartheta^*(b)$ is the informed fraction (Definition 3). Choosing $b = 0$ means keeping the market closed and reporting the prior, with no trading loss or capital charge.

The next proposition gives an upper bound on optimal liquidity and characterizes the market maker's value in sufficiently deep markets.

**Proposition 11** (The market maker's value)**.** *Under Assumption 1, let $V$ be given by* (25)*.*

*(i) Define*

$$\bar{b} := \frac{1}{c_0}\left[-\lambda \mathcal{S}(\delta_0) + \left(1 + \frac{\tau\bar{\rho}}{2\sigma_\xi}\right)\left(\frac{\tau\bar{\rho}^2}{2} + \sqrt{\tau^2\bar{\rho}^2 + \sigma_\xi^2}\right)\right].$$

[19] If users instead infer the public posterior from the price, the relevant accuracy gain is $\mathcal{S}(P_U(\bar{\delta})) - \mathcal{S}(\delta_0)$. At a fixed informed fraction, this gain does not depend on liquidity (Theorem 1(iii)); liquidity affects it only through information acquisition.

[20] The weight $\lambda$ can also represent the payment rate under a contract that rewards improvements in the execution price's realized logarithmic score over the prior. The expected payment is then $\lambda[\mathcal{S}(\bar{\delta}) - \mathcal{S}(\delta_0)]$.

*For every $b \geq \bar{b}$, $V(b) < V(0) = 0$, regardless of the informed fraction.*

*(ii) Under the hypotheses of Corollary 1, along its acquisition equilibrium $\vartheta^*(b)$, for all sufficiently large $b$: $\mathcal{S}(P_U(\bar{\delta}))$ is increasing in $b$, $\mathcal{S}(\bar{\delta})$ and $V(b)$ are strictly decreasing, and $\mathbb{E}[\mathscr{L}]$ is strictly increasing.*

*Proof.* Appendix A.26.

Capital costs limit the liquidity the market maker is willing to supply. The accuracy gain cannot exceed the improvement from the prior to a perfect forecast, and expected trading profits, $-\mathbb{E}[\mathscr{L}]$, are bounded above uniformly across liquidity levels and informed fractions. The required reserve grows in proportion to liquidity, so its funding cost eventually exceeds the combined value of forecast accuracy and trading profits.

In sufficiently deep markets, greater liquidity encourages information acquisition, making the price more informative and improving the public posterior's expected score. Greater liquidity also brings the execution price closer to the prior. This movement toward the prior outweighs the improvement from additional acquisition, reducing the accuracy of the displayed forecast. Both the increase in liquidity at a fixed informed fraction and the induced acquisition raise the market maker's expected trading loss $\mathbb{E}[\mathscr{L}]$. Lower forecast accuracy, higher expected trading losses, and a larger capital charge therefore reduce its value.

# 7. Conclusion

We study costly information acquisition in a rational expectations equilibrium with an automated market maker using the logarithmic market scoring rule. All orders execute in a single batch at a common price per share. As in GS, noise trading prevents full revelation and preserves the incentive to acquire information. When informed and uninformed traders

coexist, the signal's private value equals its acquisition cost. At a fixed informed fraction, liquidity changes price sensitivity without changing the information revealed by the price. Liquidity therefore affects price informativeness only through information acquisition.

The execution price determines trading payments, whereas the public posterior is the event probability conditional on that price. At a fixed positive informed fraction, the execution price moves farther from the prior than the public posterior does in the low-liquidity limit, but less in sufficiently deep markets. Users who interpret the displayed price directly as an event probability therefore overstate the information's implications in shallow markets and understate them in deep markets. The execution price's forecast accuracy is highest at an intermediate liquidity.

Noise trading preserves private information but can also reduce its value by moving the execution price toward zero or one. At any fixed finite liquidity and positive informed fraction, the signal's value vanishes at both extremes of noise-trading volatility. With negligible noise, the price reveals the signal; with extreme noise, the price approaches the payoff bounds, leaving little gain from acquiring information even though the signal remains largely private.

The market maker's expected trading profit also depends on liquidity. At a fixed positive informed fraction, it earns an expected profit in sufficiently shallow markets, where noise traders' losses exceed informed and uninformed traders' trading profits. It incurs an expected loss in sufficiently deep markets. As liquidity tends to infinity, noise traders' expected losses vanish, and the market maker finances the expected trading profits of both informed traders and uninformed traders who learn from the price.

The market maker chooses liquidity by balancing the value of forecast accuracy against its expected trading loss and the capital cost of reserves. Along an interior acquisition equilibrium in a sufficiently deep market, greater liquidity encourages acquisition and improves the public posterior's accuracy. Yet the execution price moves closer to the prior and be-

comes less accurate as a forecast. Greater liquidity also raises expected trading losses and capital costs, reducing the market maker's value.

Our model combines a single trading batch, a common signal, and liquidity chosen before information acquisition. Comparing batch trading with sequential execution or trading that resumes at the terminal quote would clarify how execution rules affect information acquisition and revelation. Allowing traders to acquire distinct signals would permit an analysis of how liquidity affects the aggregation of dispersed information. Allowing the market maker to adjust liquidity as orders arrive would introduce a dynamic liquidity choice. We leave these extensions to future research.

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# A. Omitted Proofs

This appendix collects the proofs, in the order the body states the results. Two facts about the log-partition $\mathscr{A}(\eta) = \log(1 + e^{\eta})$ are used throughout. First, $\mathscr{A}'(\eta) = \sigma(\eta)$, $\eta - \mathscr{A}(\eta) = \log \sigma(\eta)$, and $-\mathscr{A}(\eta) = \log(1 - \sigma(\eta))$, all by direct computation. Second, the Bernoulli relative entropy is the Bregman divergence of $\mathscr{A}$: for $\eta_1, \eta_2 \in \mathbb{R}$,

$$D(\eta_1, \eta_2) \;:=\; \mathscr{A}(\eta_1) - \mathscr{A}(\eta_2) - \sigma(\eta_2)(\eta_1 - \eta_2) \;=\; \mathrm{KL}(\sigma(\eta_2) \,\|\, \sigma(\eta_1)). \tag{26}$$

To see this, write $p = \sigma(\eta_2)$ and $q = \sigma(\eta_1)$; then $\mathrm{KL}(p \,\|\, q) = p(\operatorname{logit} p - \operatorname{logit} q) + \log \frac{1-p}{1-q} = \sigma(\eta_2)(\eta_2 - \eta_1) - \mathscr{A}(\eta_2) + \mathscr{A}(\eta_1)$, using $\operatorname{logit} \sigma(\eta) = \eta$ and $\log(1 - \sigma(\eta)) = -\mathscr{A}(\eta)$. Since $\mathscr{A}' = \sigma$, (26) also reads $D(\eta_1, \eta_2) = \int_{\eta_2}^{\eta_1} (\sigma(u) - \sigma(\eta_2))\, du$, and because $\sigma' = \sigma(1 - \sigma) \le 1/4$ the integrand is bounded by $|u - \eta_2|/4$, so that

$$0 \;\le\; D(\eta_1, \eta_2) \;\le\; \frac{1}{8}(\eta_1 - \eta_2)^2. \tag{27}$$

For a random variable or random vector $X$, $\mathscr{F}(X)$ denotes the $\sigma$-field that $X$ generates.

## A.1 Proof of Lemma 1

Write $\eta_0 := \operatorname{logit} \delta_0$ and $\eta^* := \eta_0 + x/b = \operatorname{logit} \delta^*$, the latter by the definition of the terminal quote in Definition 1.

(i) By (2), the marginal quote at the inventory $b\eta_0 + sx$ is $\sigma(\eta_0 + sx/b)$, so (3) reads $\bar{\delta} = \int_0^1 \sigma(\eta_0 + s(\eta^* - \eta_0))\, ds$. If $x = 0$ then $\eta^* = \eta_0$, the integrand is the constant $\sigma(\eta_0) = \delta_0$, and $\bar{\delta} = \delta_0$. If $x \neq 0$, substitute $u = \eta_0 + s(\eta^* - \eta_0)$, $du = (\eta^* - \eta_0)\, ds$:

$$\bar{\delta} \;=\; \frac{1}{\eta^* - \eta_0} \int_{\eta_0}^{\eta^*} \sigma(u)\, du \;=\; \frac{\mathscr{A}(\eta^*) - \mathscr{A}(\eta_0)}{\eta^* - \eta_0},$$

the second equality because $\mathscr{A}' = \sigma$. This is (4).

(ii) Trader $i$'s payment is $x_i(\bar{\delta})\bar{\delta}$ and the noise order's is $\xi\bar{\delta}$ (Definition 1(ii)); summing and using clearing (Definition 1(iii)), total payments equal $x\bar{\delta}$. If $x \neq 0$, by (i) and (2), $x\bar{\delta} = b(\eta^* - \eta_0)\bar{\delta} = b[\mathscr{A}(\eta^*) - \mathscr{A}(\eta_0)] = C_b(b\eta_0 + x) - C_b(b\eta_0) = \Delta C$; if $x = 0$ both sides vanish. The market maker receives $\Delta C$ in the batch and, having sold $x$ net YES shares, pays $xB$ at settlement, for the net cash flow $\Delta C - xB$.

(iii) Let $x \neq 0$ and suppose $\eta^* > \eta_0$. For $s \in (0, 1)$ the argument $\eta_0 + s(\eta^* - \eta_0)$ lies strictly between $\eta_0$ and $\eta^*$, and $\sigma$ is strictly increasing, so $\delta_0 = \sigma(\eta_0) < \sigma(\eta_0 + s(\eta^* - \eta_0)) < \sigma(\eta^*) = \delta^*$. Integrating a continuous function that lies strictly between two constants on $(0, 1)$ gives $\delta_0 < \bar{\delta} < \delta^*$. The case $\eta^* < \eta_0$ is symmetric. For monotonicity, view $\bar{\delta}$ as the function $g(\eta^*) := \int_0^1 \sigma(\eta_0 + s(\eta^* - \eta_0))\, ds$ of the terminal log-odds, defined for every $\eta^* \in \mathbb{R}$ and equal

to (4) by (i). The integrand is smooth in $\eta^*$ with derivatives bounded on bounded sets, so $g$ is smooth and may be differentiated under the integral sign:

$$g'(\eta^*) \;=\; \int_0^1 s\,\sigma'(\eta_0 + s(\eta^* - \eta_0))\,ds \;>\; 0,$$

because $\sigma' > 0$. Since $\eta^* = \operatorname{logit}\delta^*$ is a smooth strictly increasing function of $\delta^* \in (0,1)$, $\bar{\delta}$ is a smooth strictly increasing function of the terminal quote.

(iv) By $\eta - \mathscr{A}(\eta) = \log\sigma(\eta)$, in the state $B = 1$,

$$x - \Delta C \;=\; b[(\eta^* - \eta_0) - \mathscr{A}(\eta^*) + \mathscr{A}(\eta_0)] \;=\; b[\log\delta^* - \log\delta_0] \;<\; -b\log\delta_0,$$

the inequality because $\delta^* < 1$. By $-\mathscr{A}(\eta) = \log(1 - \sigma(\eta))$, in the state $B = 0$,

$$-\Delta C \;=\; b[\log(1-\delta^*) - \log(1-\delta_0)] \;<\; -b\log(1-\delta_0).$$

Hence $xB - \Delta C \le b\max\{-\log\delta_0, -\log(1-\delta_0)\} = bM(\delta_0)$ in both states. Q.E.D.

## A.2 Proof of Lemma 2

Write $p$ for $\bar{\delta}$ inside this proof. The trader's expected utility from the net order $x$ is

$$U(x) \;=\; -e^{-W/\tau}\Big[P\,e^{-x(1-p)/\tau} + (1-P)\,e^{xp/\tau}\Big].$$

Its second derivative, $U''(x) = -e^{-W/\tau}\tau^{-2}[P(1-p)^2e^{-x(1-p)/\tau} + (1-P)p^2e^{xp/\tau}]$, is strictly negative because $p \in (0,1)$, so $U$ is strictly concave. As $x \to +\infty$ the second exponential diverges ($p > 0$) and as $x \to -\infty$ the first does ($p < 1$), so $U(x) \to -\infty$ in both tails. A strictly concave function on $\mathbb{R}$ that tends to $-\infty$ in both directions attains a unique maximum, at the unique root of the first-order condition

$$P(1-p)\,e^{-x(1-p)/\tau} \;=\; (1-P)\,p\,e^{xp/\tau};$$

taking logarithms, $\operatorname{logit}P - \operatorname{logit}p = x/\tau$, which is (5).

For the certainty equivalent, factor $e^{xp/\tau}$ out of the bracket: $-U(x)e^{W/\tau} = e^{xp/\tau}[Pe^{-x/\tau} + (1-P)]$. At $x^* := x(P,p)$, $e^{-x^*/\tau} = e^{\operatorname{logit}p - \operatorname{logit}P}$, so $Pe^{-x^*/\tau} + (1-P) = (1-P)[\dfrac{p}{1-p} + 1] = \dfrac{1-P}{1-p}$, and the certainty equivalent $-\tau\log(-U(x^*))$ equals

$$W - x^*p - \tau\log\frac{1-P}{1-p} \;=\; W + \tau\Big[p\,(\operatorname{logit}p - \operatorname{logit}P) + \log\frac{1-p}{1-P}\Big] \;=\; W + \tau\,\mathrm{KL}(p\,\|\,P),$$

the last equality being the identity $\mathrm{KL}(p \,\|\, P) = p(\operatorname{logit} p - \operatorname{logit} P) + \log \frac{1-p}{1-P}$ obtained by expanding the definition of $\mathrm{KL}(\cdot \,\|\, \cdot)$. Q.E.D.

## A.3 Proof of Lemma 3

Write $\eta_0 := \operatorname{logit} \delta_0$, $\eta^* := \operatorname{logit} \delta^*$, and $s := \int \operatorname{logit} P_i \, di + \xi/\tau$, which is finite by hypothesis. By Definition 1, the batch clears at $(\bar{\delta}, x)$ if $\bar{\delta}$ is given by (3), which by Lemma 1(i) is the function $g(\eta^*)$ of Appendix A.1 evaluated at $\eta^* = \eta_0 + x/b$, and $x = \int x(P_i, \bar{\delta}) \, di + \xi = \tau(s - \operatorname{logit} \bar{\delta})$ by (5). Substituting $x = b(\eta^* - \eta_0)$, the second condition reads

$$\Psi_b(\eta^*) \;:=\; b(\eta^* - \eta_0) + \tau(\operatorname{logit} g(\eta^*) - \eta_0) \;=\; \tau(s - \eta_0), \tag{28}$$

and conversely at every root $\eta^*$ of (28) the batch clears at $(g(\eta^*),\, b(\eta^* - \eta_0))$. So the pairs at which the batch clears correspond one to one to roots of (28).

$\Psi_b$ is continuous and strictly increasing on $\mathbb{R}$: $g$ is smooth and strictly increasing with values in $(0,1)$ (Appendix A.1(iii)), logit is strictly increasing, and $b > 0$. As $\eta^* \to -\infty$, for every $s' \in (0,1]$ the integrand $\sigma(\eta_0 + s'(\eta^* - \eta_0))$ of $g$ tends to 0 and is bounded by 1, so $g(\eta^*) \to 0$ by dominated convergence and $\operatorname{logit} g(\eta^*) \to -\infty$; together with $b(\eta^* - \eta_0) \to -\infty$, $\Psi_b(\eta^*) \to -\infty$. Symmetrically $g(\eta^*) \to 1$ and $\Psi_b(\eta^*) \to +\infty$ as $\eta^* \to +\infty$. A continuous strictly increasing function from $\mathbb{R}$ onto $\mathbb{R}$ has exactly one root of (28) for each $s$, by the intermediate value theorem and strict monotonicity; hence the batch clears at exactly one pair.

The root $\eta^* = \Psi_b^{-1}(\tau(s - \eta_0))$ depends on the posteriors and the noise order only through $s$, and it is strictly increasing in $s$ because $\Psi_b^{-1}$ is strictly increasing. Hence $\delta^* = \sigma(\eta^*)$ and $\bar{\delta} = g(\eta^*)$ are strictly increasing functions of $s$. Q.E.D.

## A.4 The informed trader learns nothing from the price

**Lemma 5.** *Let $(L, \xi) \mapsto \bar{\delta}(L, \xi)$ be any measurable price map. Then $\Pr(B = 1 \mid S, \bar{\delta}) = \Pr(B = 1 \mid S) = P_I$ almost surely.*

*Proof.* Since $L$ is a function of $S$, the price $\bar{\delta} = \bar{\delta}(L, \xi)$ is measurable with respect to $\mathcal{F}(S, \xi)$, so it suffices to show $\Pr(B = 1 \mid S, \xi) = P_I$; the claim then follows from the tower property, $\Pr(B = 1 \mid S, \bar{\delta}) = \mathbb{E}[\Pr(B = 1 \mid S, \xi) \mid S, \bar{\delta}] = \mathbb{E}[P_I \mid S, \bar{\delta}] = P_I$, the last equality because $P_I$ is a function of $S$. For bounded measurable $f$ and $h$,

$$\mathbb{E}[\mathbf{1}\{B = 1\} f(S) h(\xi)] = \mathbb{E}[\mathbf{1}\{B = 1\} f(S)] \mathbb{E}[h(\xi)] = \mathbb{E}[P_I f(S)] \mathbb{E}[h(\xi)] = \mathbb{E}[P_I f(S) h(\xi)],$$

where the first and third equalities use the independence of $\xi$ from $(B, S)$ and the second is the definition of $P_I = \Pr(B = 1 \mid S)$. Products $f(S)h(\xi)$ form a multiplicative class

generating $\mathscr{F}(S,\xi)$, so the identity extends to every bounded $\mathscr{F}(S,\xi)$-measurable function by the monotone class theorem, which is the statement $\Pr(B=1 \mid S,\xi) = P_I$. Q.E.D.

## A.5 Proof of Lemma 4

Fix an equilibrium and a state $(L,\xi)$, write $\bar{\delta} := \bar{\delta}(L,\xi)$, and let $x$ be a total net order with which the batch clears at $\bar{\delta}$ (Definition 2(iii)), with terminal log-odds $\eta^* = \operatorname{logit}\delta^* = \eta_0 + x/b$, where $\eta_0 := \operatorname{logit}\delta_0$ (Definition 1). By Definition 2(ii) and (iii), the total net order is the sum of the schedules at the execution price,

$$\begin{aligned} x &= \vartheta\, x(P_I,\bar{\delta}) + (1-\vartheta)\, x(P_U(\bar{\delta}),\bar{\delta}) + \xi \\ &= \vartheta\tau(\eta_0 + L - \operatorname{logit}\bar{\delta}) + (1-\vartheta)\tau(\operatorname{logit}P_U(\bar{\delta}) - \operatorname{logit}\bar{\delta}) + \xi, \end{aligned}$$

using (5) and $\operatorname{logit}P_I = \eta_0 + L$ from (1). Substituting $x = b(\eta^* - \eta_0)$ and collecting the terms in $\operatorname{logit}\bar{\delta}$, which sum to $-\tau\operatorname{logit}\bar{\delta}$,

$$b(\eta^* - \eta_0) + \tau(\operatorname{logit}\bar{\delta} - \eta_0) \;=\; \vartheta\tau L + \xi + (1-\vartheta)\tau(\operatorname{logit}P_U(\bar{\delta}) - \eta_0),$$

where the left side is $\Psi_b(\eta^*)$ by (9), since $\bar{\delta} = \bar{\delta}(\delta^*)$ by Lemma 1(i). Rearranging gives (10). Finally, the right side of (10) is a function of $\bar{\delta}$ alone: $\eta^*$ is a function of $\bar{\delta}$ because $\delta^* \mapsto \bar{\delta}(\delta^*)$ is strictly increasing (Lemma 1(iii)), so every term on the right side is a function of $\bar{\delta}$. Q.E.D.

## A.6 Proof of Theorem 1

Throughout, $\eta_0 := \operatorname{logit}\delta_0$, $g$ is the map of Appendix A.1 from the terminal log-odds to the execution price, so that $\bar{\delta}(\delta^*) = g(\operatorname{logit}\delta^*)$ and $\Psi_b(\eta) = b(\eta-\eta_0) + \tau(\operatorname{logit}g(\eta) - \eta_0)$, and $v_\vartheta = \vartheta^2\tau^2\bar{\rho}^2 + \sigma_\xi^2$ as in (11). Four facts are used repeatedly. (a) $g$ is a smooth, strictly increasing bijection from $\mathbb{R}$ onto $(0,1)$: strict monotonicity and smoothness are Lemma 1(iii), and the limits $g(\eta)\to 0$ as $\eta\to-\infty$ and $g(\eta)\to 1$ as $\eta\to+\infty$ are shown in Appendix A.3; its inverse is therefore continuous. (b) $\Psi_b$ is a continuous, strictly increasing bijection of $\mathbb{R}$ (Appendix A.3), so $\Psi_b^{-1}$ is continuous and strictly increasing; it is smooth because $\Psi_b$ is smooth with $\Psi_b'(\eta) = b + \tau g'(\eta)/[g(\eta)(1-g(\eta))] > 0$. (c) The state $(L,\xi)$ has a positive density on $\mathbb{R}^2$: under Assumption 1 the distribution of $L$ is a mixture of two normal distributions and $\xi$ is normal and independent of $L$. (d) Conditional on $B$, the statistic $\Xi_\vartheta = \vartheta\tau L + \xi$ is normal with mean $\mu_B := (2B-1)\vartheta\tau\bar{\rho}^2/2$ and variance $v_\vartheta$, as a sum of independent normals, and by Bayes' rule

$$\operatorname{logit}\Pr(B=1 \mid \Xi_\vartheta = u) \;=\; \eta_0 + \log\frac{\phi((u-\mu_1)/\sqrt{v_\vartheta})}{\phi((u-\mu_0)/\sqrt{v_\vartheta})} \;=\; \eta_0 + \frac{\mu_1-\mu_0}{v_\vartheta}\,u \;=\; \eta_0 + \beta_\vartheta u, \quad (29)$$

because $\mu_0 = -\mu_1$ makes the quadratic terms cancel and $(\mu_1 - \mu_0)/v_\vartheta = \vartheta\tau\bar{\rho}^2/v_\vartheta = \beta_\vartheta$.

*Step 1: a continuous price map depends on the state only through $\Xi_\vartheta$.* Let $\bar{\delta}(\cdot,\cdot)$ be the price map of a continuous equilibrium. By Lemma 4 there is a function $\mathfrak{u}$ on $(0,1)$, namely the right side of (10) read as a function of the price, with $\Xi_\vartheta(z) = \mathfrak{u}(\bar{\delta}(z))$ for every state $z = (L,\xi)$. Parametrize the states by $z(u,t) := (t,\, u - \vartheta\tau t)$, so that $\Xi_\vartheta(z(u,t)) = u$ and, for fixed $u$, $t \mapsto z(u,t)$ runs over the line $\{\Xi_\vartheta = u\}$; the map $\tilde{p}(u,t) := \bar{\delta}(z(u,t))$ is continuous on $\mathbb{R}^2$. Suppose $\tilde{p}(u,t_1) < \tilde{p}(u,t_2)$ for some $u$ and $t_1, t_2$, and let $m$ be the midpoint of the two values. By continuity of $\tilde{p}$ at $(u,t_1)$ and at $(u,t_2)$ there is $u' \neq u$ with $\tilde{p}(u',t_1) < m < \tilde{p}(u',t_2)$. By the intermediate value theorem applied to the continuous functions $t \mapsto \tilde{p}(u,t)$ and $t \mapsto \tilde{p}(u',t)$ there are $t^*$ and $t'$ with $\tilde{p}(u,t^*) = m = \tilde{p}(u',t')$: two states with the same price and different values of $\Xi_\vartheta$, contradicting $\Xi_\vartheta = \mathfrak{u}(\bar{\delta})$. Hence $\tilde{p}(u,\cdot)$ is constant for every $u$, and $\bar{\delta}(z) = h(\Xi_\vartheta(z))$ with $h(u) := \tilde{p}(u,0)$, a continuous function of $u$. Moreover $\mathfrak{u}(h(u)) = u$ for every $u$, so $h$ is injective.

*Step 2: uniqueness.* Continue with a continuous equilibrium. By Step 1, $\mathscr{F}(\bar{\delta}) \subseteq \mathscr{F}(\Xi_\vartheta)$, and $\Xi_\vartheta = \mathfrak{u}(\bar{\delta})$ gives the reverse inclusion, so $\mathscr{F}(\bar{\delta}) = \mathscr{F}(\Xi_\vartheta)$ and $\Pr(B = 1 \mid \bar{\delta}) = \Pr(B = 1 \mid \Xi_\vartheta)$ almost surely. By Definition 2(i) and (29), $\operatorname{logit} P_U(\bar{\delta}) = \eta_0 + \beta_\vartheta \Xi_\vartheta$ almost surely. Substituting into (10),

$$\Psi_b(\eta^*) \;=\; \Xi_\vartheta + (1-\vartheta)\tau\beta_\vartheta\Xi_\vartheta \;=\; \varpi_\vartheta\Xi_\vartheta \qquad \text{almost surely,}$$

where $\eta^* = g^{-1}(\bar{\delta}(z))$ is the terminal log-odds in the state $z$. Both sides are continuous functions of $z$, the left by (a), (b), and the continuity of the price map, and two continuous functions that agree almost surely under a distribution with a positive density on $\mathbb{R}^2$ agree everywhere, their difference being continuous and vanishing on a dense set. Hence $\eta^* = \Psi_b^{-1}(\varpi_\vartheta\Xi_\vartheta)$ and $\bar{\delta} = g(\eta^*)$ in every state, which is (12): any two continuous equilibria have the same price map. If $\vartheta < 1$, reading (10) in every state with this price map gives $(1-\vartheta)\tau(\operatorname{logit} P_U(\bar{\delta}) - \eta_0) = \Psi_b(\eta^*) - \Xi_\vartheta = (\varpi_\vartheta - 1)\Xi_\vartheta$, so $\operatorname{logit} P_U(\bar{\delta}) = \eta_0 + \beta_\vartheta\Xi_\vartheta$ in every state; since $u \mapsto g(\Psi_b^{-1}(\varpi_\vartheta u))$ maps $\mathbb{R}$ onto $(0,1)$ by (a) and (b), this determines $P_U$ at every price. (If $\vartheta = 1$ there are no uninformed traders and $P_U$ does not enter the equilibrium.)

*Step 3: existence.* Define the price map and the posterior by

$$\bar{\delta}(z) := h(\Xi_\vartheta(z)), \qquad h(u) := g(\Psi_b^{-1}(\varpi_\vartheta u)), \qquad P_U(p) := \sigma(\eta_0 + \beta_\vartheta h^{-1}(p)) \quad \text{for } p \in (0,1),$$

which is (12). Since $\varpi_\vartheta > 0$, $h$ is a continuous strictly increasing bijection of $\mathbb{R}$ onto $(0,1)$ by (a) and (b), so $\bar{\delta}$ is continuous on $\mathbb{R}^2$ and $\mathscr{F}(\bar{\delta}) = \mathscr{F}(\Xi_\vartheta)$. Condition (i): $\Pr(B = 1 \mid \bar{\delta} = p) = \Pr(B = 1 \mid \Xi_\vartheta = h^{-1}(p)) = \sigma(\eta_0 + \beta_\vartheta h^{-1}(p)) = P_U(p)$ by (29), for every $p$. Condition (ii) is the specification of the schedules. Condition (iii): fix a state $z$ with $u = \Xi_\vartheta(z)$. By Definition 1, the batch clears with terminal log-odds $\eta$ if, at the execution price $g(\eta)$, the schedules and the noise order sum to $b(\eta - \eta_0)$; the computation of Appendix A.5 shows this

holds if and only if

$$\Psi_b(\eta) \;=\; u+(1-\vartheta)\tau(\operatorname{logit} P_U(g(\eta))-\eta_0) \;=\; u+(1-\vartheta)\tau\beta_\vartheta\, h^{-1}(g(\eta)) \;=\; u+\frac{\varpi_\vartheta-1}{\varpi_\vartheta}\,\Psi_b(\eta),$$

the last equality because $h^{-1}(g(\eta)) = \Psi_b(\eta)/\varpi_\vartheta$ by the definition of $h$. This is $\Psi_b(\eta) = \varpi_\vartheta u$, whose unique solution is $\eta = \Psi_b^{-1}(\varpi_\vartheta u)$, with execution price $h(u) = \bar\delta(z)$. So in every state the batch clears at exactly one pair, whose execution price is $\bar\delta(z)$. This proves (i).

*Step 4: parts (ii) and (iii).* In the equilibrium, $\operatorname{logit}\delta^* = \Psi_b^{-1}(\varpi_\vartheta \Xi_\vartheta)$ is smooth and strictly increasing in $\Xi_\vartheta$ by (b), and $\bar\delta = g(\operatorname{logit}\delta^*)$ is smooth and strictly increasing in it by (a). A strictly increasing continuous function of a real random variable generates the same $\sigma$-field as the variable, since it is a homeomorphism onto its image; hence $\mathscr{F}(\delta^*) = \mathscr{F}(\bar\delta) = \mathscr{F}(\Xi_\vartheta)$, which is (ii). The statistic $\Xi_\vartheta$ does not involve $b$, which is (iii). Q.E.D.

## A.7 The equilibrium under general primitives

Theorem 1 uses Assumption 1 only through facts (c) and (d) of Appendix A.6, the positive density of the state and the linear posterior (29). The theorem of this section replaces them by regularity conditions on the primitives. Throughout, $\eta_0$, $g$, and $\Psi_b$ are as in Appendix A.6, and facts (a) and (b) there, which do not involve the distributions of $L$ and $\xi$, are used as they stand. Under the hypotheses of the theorem, conditional on the event the statistic $\Xi_\vartheta$ has a density $f_B$ that is positive and continuous on $\mathbb{R}$ (fact (d′) in the proof), and for $u \in \mathbb{R}$ we write

$$r_\vartheta(u) \;:=\; \eta_0 + \log\frac{f_1(u)}{f_0(u)}, \qquad G_\vartheta(u) \;:=\; u + (1-\vartheta)\tau(r_\vartheta(u)-\eta_0), \tag{30}$$

so that $r_\vartheta$ is a continuous version of $\operatorname{logit}\Pr(B=1 \mid \Xi_\vartheta = u)$ by Bayes' rule and $G_\vartheta$ is the right side of (10) solved for $\Psi_b(\operatorname{logit}\delta^*)$ when the uninformed posterior is formed from $\Xi_\vartheta$. A density $f$ on $\mathbb{R}$ is *log-concave* if $\log f$ is concave; the normal, logistic, and Laplace densities are log-concave, and a log-concave density that is positive on $\mathbb{R}$ is continuous and bounded.[21]

**Theorem 7** (The equilibrium under general primitives)**.** *Fix $\vartheta \in [0,1]$ and $b > 0$. Suppose that the distribution of $L$ has support $\mathbb{R}$ and that $\xi$ has a density $f_\xi$ that is positive, bounded, and continuous on $\mathbb{R}$. Then:*

*(i) In every continuous equilibrium, in every state,*

$$\operatorname{logit}\delta^* \;=\; \Psi_b^{-1}(G_\vartheta(\Xi_\vartheta)), \qquad \bar\delta \;=\; \bar\delta(\delta^*), \tag{31}$$

[21] A concave function that is finite on $\mathbb{R}$ is continuous. If $\psi = \log f$ were unbounded above, there would be points $x_n$ with $\psi(x_n) \geq \psi(0)$ and $|x_n| \to \infty$, and by concavity $\psi \geq \psi(0)$ on the segment between 0 and each $x_n$, hence on a half-line, contradicting the integrability of $f = e^\psi$.

*and, if* $\vartheta < 1$, $\operatorname{logit} P_U(\bar{\delta}) = r_\vartheta(\Xi_\vartheta)$.

*(ii) A continuous equilibrium exists if and only if* $G_\vartheta$ *is injective. In it,* $\delta^*$ *and* $\bar{\delta}$ *are continuous, strictly monotone functions of* $\Xi_\vartheta$*, and* $\mathscr{F}(\bar{\delta}) = \mathscr{F}(\delta^*) = \mathscr{F}(\Xi_\vartheta)$*, which does not depend on* $b$*.*

*(iii) If* $f_\xi$ *is log-concave, then* $r_\vartheta$ *is nondecreasing,* $G_\vartheta$ *is a strictly increasing bijection of* $\mathbb{R}$*, a continuous equilibrium exists, and in it* $\delta^*$ *and* $\bar{\delta}$ *are strictly increasing functions of* $\Xi_\vartheta$*.*

Under Assumption 1, $r_\vartheta(u) = \eta_0 + \beta_\vartheta u$ and $G_\vartheta(u) = \varpi_\vartheta u$ by (29), and the theorem reduces to Theorem 1. The full-support condition on $L$ makes $\mathbb{R}^2$, the domain of the price map in Definition 2, the support of the state, so that the continuity restriction binds only where states occur; the proof does not otherwise use it.[22]

*Proof.* Four facts replace (c) and (d) of Appendix A.6.

(c′) *The likelihood ratio of the signal.* Write $P_L$ for the distribution of $L$ and $P_{L|B}$ for its conditional distribution given $B$. Then $P_{L|1}(dl) = e^l\, P_{L|0}(dl)$. Indeed, for a Borel set $A \subseteq \mathbb{R}$, since $L$ is a function of $S$, the tower property and (1) give $\Pr(B = 1, L \in A) = \mathbb{E}[\mathbf{1}\{L \in A\} \Pr(B = 1 \mid S)] = \mathbb{E}[\mathbf{1}\{L \in A\}\sigma(\eta_0 + L)]$ and likewise $\Pr(B = 0, L \in A) = \mathbb{E}[\mathbf{1}\{L \in A\}(1 - \sigma(\eta_0 + L))]$. Dividing by $\delta_0 = \sigma(\eta_0)$ and by $1 - \delta_0$,

$$P_{L|1}(dl) = \frac{\sigma(\eta_0 + l)}{\sigma(\eta_0)}\, P_L(dl), \qquad P_{L|0}(dl) = \frac{1 - \sigma(\eta_0 + l)}{1 - \sigma(\eta_0)}\, P_L(dl),$$

and the ratio of the two densities is $\frac{\sigma(\eta_0+l)}{1-\sigma(\eta_0+l)} \cdot \frac{1-\sigma(\eta_0)}{\sigma(\eta_0)} = e^{\eta_0+l}e^{-\eta_0} = e^l$, using $\sigma(\eta)/(1-\sigma(\eta)) = e^\eta$.

(d′) *The conditional densities of the statistic.* Since $\xi$ is independent of $(B, S)$, it is independent of $(B, L)$, so conditional on $B$ the statistic $\Xi_\vartheta = \vartheta\tau L + \xi$ has the density

$$f_B(u) \;=\; \int f_\xi(u - \vartheta\tau l)\, P_{L|B}(dl), \qquad u \in \mathbb{R},$$

by Fubini's theorem. It is positive because $f_\xi$ is, and it is continuous because $f_\xi$ is continuous and bounded: for $u_n \to u$ the integrands converge pointwise and are dominated by the constant $\sup f_\xi$, which is integrable against the probability $P_{L|B}$, so $f_B(u_n) \to f_B(u)$ by dominated convergence. By Bayes' rule, $\Pr(B = 1 \mid \Xi_\vartheta) = \delta_0 f_1(\Xi_\vartheta)/[\delta_0 f_1(\Xi_\vartheta) + (1 - \delta_0) f_0(\Xi_\vartheta)] = \sigma(r_\vartheta(\Xi_\vartheta))$ almost surely, so $r_\vartheta$ in (30) is a continuous version of the conditional

[22] For a signal whose log-likelihood ratio is supported on an interval $I$, the same proof applies with $\mathbb{R}^2$ replaced by $I \times \mathbb{R}$ throughout Definition 2: Step 1 of Appendix A.6 runs along the segments $\{\Xi_\vartheta = u\} \cap (I \times \mathbb{R})$, and $\Xi_\vartheta$ still has a positive density on $\mathbb{R}$ because $\xi$ does.

log-odds, $G_\vartheta$ is continuous, and $\Xi_\vartheta$ has the positive density $\delta_0 f_1 + (1-\delta_0) f_0$ on $\mathbb{R}$. In particular any Borel set that is null under the distribution of $\Xi_\vartheta$ is Lebesgue-null.

(e′) *Monotone likelihood ratio under log-concave noise.* Suppose $f_\xi$ is log-concave and write $\psi := \log f_\xi$, a concave function that is finite on $\mathbb{R}$ because $f_\xi > 0$. We show that $f_1/f_0$, hence $r_\vartheta$, is nondecreasing. Fix $u < u'$ and put $s := u' - u > 0$. For $a \le a'$,

$$\frac{\psi(a+s)-\psi(a)}{s} \;\ge\; \frac{\psi(a'+s)-\psi(a)}{a'+s-a} \;\ge\; \frac{\psi(a'+s)-\psi(a')}{s},$$

because the slope of a chord of a concave function is nonincreasing in the position of either endpoint: the first inequality moves the right endpoint from $a+s$ to $a'+s$, the second moves the left endpoint from $a$ to $a'$. Hence $a \mapsto \psi(a+s)-\psi(a)$ is nonincreasing, and since $a(l) := u - \vartheta\tau l$ is nonincreasing in $l$, the function

$$\rho(l) \;:=\; \frac{f_\xi(u'-\vartheta\tau l)}{f_\xi(u-\vartheta\tau l)} \;=\; \exp[\psi(a(l)+s)-\psi(a(l))]$$

is nondecreasing in $l$. Let $w_u(dl) := f_\xi(u-\vartheta\tau l)P_{L|0}(dl)/f_0(u)$, a probability measure by (d′), so that by (c′)

$$\int e^l\, w_u(dl) \;=\; \frac{\int f_\xi(u-\vartheta\tau l)\, e^l\, P_{L|0}(dl)}{f_0(u)} \;=\; \frac{f_1(u)}{f_0(u)},$$

and likewise for $u'$; both are finite. Moreover $w_{u'}(dl) = \tilde\rho(l)\, w_u(dl)$ with $\tilde\rho(l) := \rho(l) f_0(u)/f_0(u')$, which is nondecreasing with $\int \tilde\rho\, dw_u = 1$. If $\tilde\rho \le 1$ everywhere, then $\int \tilde\rho\, dw_u = 1$ forces $\tilde\rho = 1$ $w_u$-almost everywhere, and $\int e^l\, w_{u'}(dl) = \int e^l\, w_u(dl)$. Otherwise let $l_0 := \inf\{l : \tilde\rho(l) > 1\}$, which is finite, since $l_0 = -\infty$ would make $\tilde\rho > 1$ everywhere and contradict $\int \tilde\rho\, dw_u = 1$. By monotonicity $\tilde\rho \le 1$ on $(-\infty, l_0)$ and $\tilde\rho \ge 1$ on $(l_0, \infty)$, so $(e^l - e^{l_0})(\tilde\rho(l)-1) \ge 0$ for every $l$, and integrating against $w_u$,

$$0 \;\le\; \int e^l \tilde\rho(l)\, w_u(dl) - \int e^l\, w_u(dl) - e^{l_0}\Big(\int \tilde\rho\, dw_u - 1\Big) \;=\; \frac{f_1(u')}{f_0(u')} - \frac{f_1(u)}{f_0(u)}.$$

Hence $f_1/f_0$ is nondecreasing.[23]

(f′) *$G_\vartheta$ is onto under log-concave noise.* By (e′), $r_\vartheta(u) \ge r_\vartheta(0)$ for $u \ge 0$ and $r_\vartheta(u) \le r_\vartheta(0)$ for $u \le 0$, so $G_\vartheta(u) \ge u + (1-\vartheta)\tau(r_\vartheta(0) - \eta_0) \to +\infty$ as $u \to +\infty$ and $G_\vartheta(u) \le u + (1-\vartheta)\tau(r_\vartheta(0)-\eta_0) \to -\infty$ as $u \to -\infty$; since $G_\vartheta$ is continuous by (d′), it maps $\mathbb{R}$ onto $\mathbb{R}$ by the intermediate value theorem.

*Step 1: part (i).* Let $\bar\delta(\cdot,\cdot)$ be the price map of a continuous equilibrium. Step 1 of

[23] The argument composes two kernels with the monotone likelihood ratio property, the signal's by (c′) and the noise's by log-concavity; Breon-Drish (2015, Proposition 8) records the general composition result, and the binary event makes the direct argument above shorter.

Appendix A.6 uses only Lemma 4 and the continuity of the price map, so it applies: $\bar{\delta}(z) = h(\Xi_\vartheta(z))$ in every state $z$, with $h$ continuous and injective on $\mathbb{R}$ and $\mathfrak{u}(h(u)) = u$ for every $u$. Hence $\mathscr{F}(\bar{\delta}) = \mathscr{F}(\Xi_\vartheta)$, so $\Pr(B = 1 \mid \bar{\delta}) = \Pr(B = 1 \mid \Xi_\vartheta) = \sigma(r_\vartheta(\Xi_\vartheta))$ almost surely by (d′), and Definition 2(i) gives $\operatorname{logit} P_U(\bar{\delta}) = r_\vartheta(\Xi_\vartheta)$ almost surely. Substituting into (10), which holds in every state, and writing $\eta^* = g^{-1}(\bar{\delta})$ for the terminal log-odds,

$$\Psi_b(g^{-1}(h(u))) \;=\; G_\vartheta(u) \qquad \text{for almost every } u \text{ under the distribution of } \Xi_\vartheta.$$

The function $u \mapsto \Psi_b(g^{-1}(h(u))) - G_\vartheta(u)$ is continuous, by (a), (b), the continuity of $h$, and (d′), and it vanishes off a set that is null under the distribution of $\Xi_\vartheta$, hence Lebesgue-null by (d′), hence off a set with empty interior; a continuous function that vanishes on a dense set vanishes everywhere. So $h(u) = g(\Psi_b^{-1}(G_\vartheta(u)))$ for every $u \in \mathbb{R}$, which with $\bar{\delta} = h(\Xi_\vartheta)$ in every state is (31). If $\vartheta < 1$, reading (10) in every state with this price map gives $(1-\vartheta)\tau(\operatorname{logit} P_U(\bar{\delta}) - \eta_0) = \Psi_b(\eta^*) - \Xi_\vartheta = G_\vartheta(\Xi_\vartheta) - \Xi_\vartheta = (1-\vartheta)\tau(r_\vartheta(\Xi_\vartheta) - \eta_0)$, so $\operatorname{logit} P_U(\bar{\delta}) = r_\vartheta(\Xi_\vartheta)$ in every state. This proves (i).

*Step 2: necessity in part (ii).* In a continuous equilibrium, $G_\vartheta = \Psi_b \circ g^{-1} \circ h$ by Step 1, a composition of injective maps; so $G_\vartheta$ is injective.

*Step 3: sufficiency in part (ii).* Suppose $G_\vartheta$ is injective. Being continuous by (d′), it is strictly monotone: a continuous function on $\mathbb{R}$ that is not strictly monotone takes some value at two points by the intermediate value theorem. Define

$$h(u) := g(\Psi_b^{-1}(G_\vartheta(u))), \qquad \bar{\delta}(z) := h(\Xi_\vartheta(z)), \qquad P_U(p) := \begin{cases} \sigma(r_\vartheta(h^{-1}(p))) & p \in J, \\ \delta_0 & p \in (0,1) \setminus J, \end{cases}$$

where $J := h(\mathbb{R})$. By (a) and (b), $h$ is continuous and strictly monotone, $J$ is an interval that contains none of its endpoints, and $h^{-1}$ is continuous on $J$; so $\bar{\delta}$ is continuous on $\mathbb{R}^2$ and $P_U$ is measurable. Condition (i) of Definition 2: $\bar{\delta} \in J$ in every state, and since $h$ is a homeomorphism of $\mathbb{R}$ onto $J$, $\mathscr{F}(\bar{\delta}) = \mathscr{F}(\Xi_\vartheta)$ and $\Pr(B = 1 \mid \bar{\delta}) = \Pr(B = 1 \mid \Xi_\vartheta) = \sigma(r_\vartheta(\Xi_\vartheta)) = \sigma(r_\vartheta(h^{-1}(\bar{\delta}))) = P_U(\bar{\delta})$ almost surely by (d′), so $P_U$ is a version of the conditional probability. Condition (ii) is the specification of the schedules. Condition (iii): fix a state $z$ with $u = \Xi_\vartheta(z)$. As in Step 3 of Appendix A.6, the batch clears with terminal log-odds $\eta$ if and only if $\Psi_b(\eta) = u + (1-\vartheta)\tau(\operatorname{logit} P_U(g(\eta)) - \eta_0)$. At $\hat{\eta} := \Psi_b^{-1}(G_\vartheta(u))$ we have $g(\hat{\eta}) = h(u) \in J$, so $\operatorname{logit} P_U(g(\hat{\eta})) = r_\vartheta(u)$ and the right side equals $G_\vartheta(u) = \Psi_b(\hat{\eta})$: the batch clears with terminal log-odds $\hat{\eta}$, at execution price $g(\hat{\eta}) = h(u) = \bar{\delta}(z)$. Hence a continuous equilibrium exists. In it, $\operatorname{logit} \delta^* = \hat{\eta} = \Psi_b^{-1}(G_\vartheta(\Xi_\vartheta))$ and $\bar{\delta} = g(\operatorname{logit} \delta^*)$ are continuous and strictly monotone in $\Xi_\vartheta$ by (a) and (b), in the direction of $G_\vartheta$, and $\mathscr{F}(\bar{\delta}) = \mathscr{F}(\delta^*) = \mathscr{F}(\Xi_\vartheta)$ as in Step 4 of Appendix A.6; $\Xi_\vartheta$ does not involve $b$. This proves (ii).

*Step 4: part (iii).* If $f_\xi$ is log-concave, $r_\vartheta$ is nondecreasing by (e′), so $G_\vartheta$ is strictly increasing, being the sum of $u$ and a nondecreasing function; it is continuous by (d′) and

onto $\mathbb{R}$ by (f′), hence a strictly increasing bijection, in particular injective. Part (ii) then gives a continuous equilibrium in which $\delta^*$ and $\bar{\delta}$ are strictly monotone in $\Xi_\vartheta$ in the direction of $G_\vartheta$, that is, strictly increasing. In this case $J = g(\Psi_b^{-1}(\mathbb{R})) = (0,1)$, so $P_U$ is determined at every price, and the batch clears at exactly one pair in every state: for any $\eta$, with $v := G_\vartheta^{-1}(\Psi_b(\eta)) = h^{-1}(g(\eta))$ the clearing condition of Step 3 reads $\Psi_b(\eta) = u + G_\vartheta(v) - v = u + \Psi_b(\eta) - v$, that is, $v = u$, whose unique solution is $\eta = \hat{\eta}$. Q.E.D.

## A.8 Proof of Proposition 1

Conditional on $B$, $\Xi_\vartheta$ is normal with mean $\mu_B$ and variance $v_\vartheta$ (fact (d) of Appendix A.6), so for $u \in \mathbb{R}$, $\Pr(\Xi_\vartheta \le u \mid B) = \Phi((u - \mu_B)/\sqrt{v_\vartheta})$.

(i) By (12) and facts (a) and (b), each of $\delta^*$, $\bar{\delta}$, and $x = b(\operatorname{logit}\delta^* - \eta_0)$ is a strictly increasing continuous function of $\Xi_\vartheta$, so each event below is the event that $\Xi_\vartheta$ lies below the corresponding threshold. For the terminal quote, $\delta^* \le d$ if and only if $\Psi_b^{-1}(\varpi_\vartheta \Xi_\vartheta) \le \operatorname{logit} d$, that is, $\Xi_\vartheta \le \Psi_b(\operatorname{logit} d)/\varpi_\vartheta$, whose conditional probability is $\Phi(z_B(\operatorname{logit} d))$ by (14). For the execution price, $\bar{\delta} \le p$ if and only if $\operatorname{logit}\delta^* \le \bar{\eta}(p)$, since $g$ is strictly increasing and $g(\bar{\eta}(p)) = p$, and the same computation gives $\Phi(z_B(\bar{\eta}(p)))$. For the net order, $x \le m$ if and only if $\operatorname{logit}\delta^* \le \eta_0 + m/b = \eta_m$, giving $\Phi(z_B(\eta_m))$. Each of these three conditional distribution functions is the composition of $\Phi$ with a smooth function of its argument, so its density is the derivative. Since $z_B'(\eta) = \Psi_b'(\eta)/(\varpi_\vartheta\sqrt{v_\vartheta})$ with $\Psi_b' > 0$ (fact (b)), the chain rule gives, for the terminal quote, $f_{\delta^*|B}(d) = \phi(z_B(\operatorname{logit} d))\, z_B'(\operatorname{logit} d)\, \frac{d}{dd}\operatorname{logit} d$ with $\frac{d}{dd}\operatorname{logit} d = 1/[d(1-d)]$; for the net order, $f_{x|B}(m) = \phi(z_B(\eta_m))\, z_B'(\eta_m)/b$; and for the execution price, $f_{\bar{\delta}|B}(p) = \phi(z_B(\bar{\eta}(p)))\, z_B'(\bar{\eta}(p))\, \bar{\eta}'(p)$ with $\bar{\eta}'(p) = 1/g'(\bar{\eta}(p))$ by the inverse function theorem, $g' > 0$ being fact (a).

(ii) Since $\Pr(B = 1) = \delta_0$, the law of total probability writes each unconditional distribution function as the mixture, with weights $\delta_0$ and $1 - \delta_0$, of the two conditional distribution functions found in (i). Differentiating, the unconditional density is the same mixture of the two conditional densities of (i); these share the factor that multiplies $\phi$, and taking it out of the mixture gives the results. Q.E.D.

## A.9 The individual orders

**Lemma 6.** *Under Assumption 1, in the equilibrium of Theorem 1, with $v_\vartheta$ as in* (11) *and $\eta^*(u) := \Psi_b^{-1}(\varpi_\vartheta u)$:*

*(i) Conditional on $\Xi_\vartheta = u$ and $B$, the informed order $x_I$ is normal with mean $\tau[\operatorname{logit}\delta_0 + \beta_\vartheta u + (2B-1)\dfrac{\bar{\rho}^2}{2}(1 - \vartheta\tau\beta_\vartheta) - \operatorname{logit} g(\eta^*(u))]$ and variance $\tau^2\bar{\rho}^2\sigma_\xi^2/v_\vartheta$; its distribution conditional on $B$ alone is the mixture of these normals over the normal distribution of $\Xi_\vartheta$ given $B$.*

*(ii) The uninformed order is the smooth function* $x_U(u) = \tau[\operatorname{logit}\delta_0 + \beta_\vartheta u - \operatorname{logit} g(\eta^*(u))]$ *of* $\Xi_\vartheta$*, with derivative*

$$x_U'(u) \;=\; \tau\Big[\beta_\vartheta - \frac{\varpi_\vartheta\,(\operatorname{logit} g)'(\eta^*(u))}{\Psi_b'(\eta^*(u))}\Big],$$

*which vanishes exactly where* $(\operatorname{logit} g)'(\eta^*(u)) = \beta_\vartheta b/(1-\vartheta\tau\beta_\vartheta)$*; at the prior,* $x_U'(0) = \tau[\beta_\vartheta - \varpi_\vartheta/(2b+\tau)]$*, which is negative if and only if* $\beta_\vartheta(2b+\vartheta\tau) < 1$*, and so for every* $b$ *small enough. At any* $q$ *that is a regular value of* $x_U$*, the conditional density of* $x_U$ *given* $B$ *is* $\sum_{u:\,x_U(u)=q} \phi((u-\mu_B)/\sqrt{v_\vartheta})/(\sqrt{v_\vartheta}\,|x_U'(u)|)$*.*

*Proof.* (i) Conditional on $B$, $(L, \Xi_\vartheta)$ is bivariate normal, with $\mathbb{E}[L \mid B] = (2B-1)\bar\rho^2/2$, $\operatorname{Var}(L \mid B) = \bar\rho^2$, $\mathbb{E}[\Xi_\vartheta \mid B] = \mu_B = (2B-1)\vartheta\tau\bar\rho^2/2$, $\operatorname{Var}(\Xi_\vartheta \mid B) = v_\vartheta$, and $\operatorname{Cov}(L, \Xi_\vartheta \mid B) = \vartheta\tau\bar\rho^2$, because $\xi$ is independent of $L$. The conditional distribution of one coordinate of a bivariate normal given the other is normal with mean $\mathbb{E}[L \mid B] + \frac{\operatorname{Cov}}{\operatorname{Var}(\Xi_\vartheta|B)}(u-\mu_B)$ and variance $\operatorname{Var}(L \mid B) - \operatorname{Cov}^2/\operatorname{Var}(\Xi_\vartheta \mid B)$. Here $\operatorname{Cov}/v_\vartheta = \beta_\vartheta$, so the mean is $(2B-1)\dfrac{\bar\rho^2}{2} + \beta_\vartheta(u-\mu_B) = \beta_\vartheta u + (2B-1)\dfrac{\bar\rho^2}{2}(1-\vartheta\tau\beta_\vartheta)$ and the variance is $\bar\rho^2 - \vartheta^2\tau^2\bar\rho^4/v_\vartheta = \bar\rho^2\sigma_\xi^2/v_\vartheta$. By (13), $x_I = \tau(\operatorname{logit}\delta_0 + L - \operatorname{logit} g(\eta^*(u)))$ is an affine function of $L$ with slope $\tau$ given $\Xi_\vartheta = u$, which gives the stated mean and variance; integrating over the conditional distribution of $\Xi_\vartheta$ gives the mixture.

(ii) The formula for $x_U$ is (13) with $\operatorname{logit}\bar\delta = \operatorname{logit} g(\eta^*(u))$. It is smooth by facts (a) and (b) of Appendix A.6, and by the chain rule and the inverse function theorem, $\frac{d}{du}\eta^*(u) = \varpi_\vartheta/\Psi_b'(\eta^*(u))$, which gives $x_U'$. Writing $H'$ for $(\operatorname{logit} g)'(\eta^*(u))$ and using $\Psi_b' = b + \tau H'$, the bracket vanishes if and only if $\beta_\vartheta(b+\tau H') = \varpi_\vartheta H'$, that is, $\beta_\vartheta b = H'(\varpi_\vartheta - \tau\beta_\vartheta) = H'(1-\vartheta\tau\beta_\vartheta)$, using (11); note $1-\vartheta\tau\beta_\vartheta = \sigma_\xi^2/v_\vartheta > 0$. At $u=0$, $\eta^*(0) = \eta_0$ because $\Psi_b(\eta_0) = 0$, and $H' = 1/2$ there: by Appendix A.1, $g'(\eta_0) = \sigma'(\eta_0)\int_0^1 s\,ds = \delta_0(1-\delta_0)/2$ and $g(\eta_0) = \delta_0$, so $(\operatorname{logit} g)'(\eta_0) = g'(\eta_0)/[\delta_0(1-\delta_0)] = 1/2$. Hence $\Psi_b'(\eta_0) = b + \tau/2$ and $x_U'(0) = \tau[\beta_\vartheta - \varpi_\vartheta/(2b+\tau)]$; this is negative if and only if $\beta_\vartheta(2b+\tau) < 1 + (1-\vartheta)\tau\beta_\vartheta$, that is, $\beta_\vartheta(2b+\vartheta\tau) < 1$, and since $\beta_\vartheta\vartheta\tau = \vartheta^2\tau^2\bar\rho^2/v_\vartheta < 1$ the inequality holds for every $b$ small enough. The density formula is the change-of-variables formula for the pushforward of the normal distribution of $\Xi_\vartheta$ given $B$ under the smooth map $x_U$, which holds at every regular value, where the preimage is a discrete set on which $x_U' \neq 0$ and the map is locally a diffeomorphism. Q.E.D.

## A.10 Proof of Proposition 2

Fix the equilibrium and write $p := \bar{\delta}$ for the execution price in a state. For an order $q$, a price $p$, and a posterior $P$, let

$$\mathscr{M}(q; p, P) \;:=\; P\, e^{-q(1-p)/\tau} + (1-P)\, e^{qp/\tau},$$

so that a trader with posterior $P$ and initial wealth $W'$ who holds the order $q$ at the price $p$ has expected utility $-e^{-W'/\tau}\mathscr{M}(q; p, P)$. By Appendix A.2, $\mathscr{M}(\cdot; p, P)$ is strictly convex, its unique minimizer is $x(P, p) = \tau(\operatorname{logit} P - \operatorname{logit} p)$, and its minimum is $e^{-\mathrm{KL}(p \,\|\, P)}$.

(i) An informed trader's posterior given her information $(S, \bar{\delta})$ is $P_I$ (Lemma 5), her wealth after paying for the signal is $W - c_I$, and her order is $x(P_I, \bar{\delta})$ (Definition 2(ii)); her expected utility conditional on $(S, \bar{\delta})$ is therefore $-e^{-(W-c_I)/\tau}e^{-D_I}$, whose certainty equivalent is $W - c_I + \tau D_I$. Taking the expectation over the state, her expected utility is $-e^{-(W-c_I)/\tau}\mathbb{E}[e^{-D_I}]$, with certainty equivalent $W - c_I - \tau \log \mathbb{E}[e^{-D_I}] = W - c_I + G_I$. The uninformed trader's information is the price, her posterior is $P_U(\bar{\delta}) = \Pr(B = 1 \mid \bar{\delta})$ almost surely (Definition 2(i)), and her order is $x(P_U(\bar{\delta}), \bar{\delta})$; the same two steps give $W + \tau D_U$ and $W + G_U$.

(ii) Write $x_U := x(P_U(\bar{\delta}), \bar{\delta})$, the uninformed trader's order, a function of the price. Since $x(P_I, \bar{\delta})$ minimizes $\mathscr{M}(\cdot; \bar{\delta}, P_I)$,

$$e^{-D_I} \;=\; \mathscr{M}(x(P_I, \bar{\delta}); \bar{\delta}, P_I) \;\le\; \mathscr{M}(x_U; \bar{\delta}, P_I) \qquad \text{in every state.} \tag{32}$$

Take the conditional expectation given $\bar{\delta}$. Since $x_U$ and $\bar{\delta}$ are functions of $\bar{\delta}$, and $\mathscr{M}$ is affine in $P$,

$$\mathbb{E}[\mathscr{M}(x_U; \bar{\delta}, P_I) \mid \bar{\delta}] \;=\; \mathscr{M}(x_U; \bar{\delta}, \mathbb{E}[P_I \mid \bar{\delta}]) \;=\; \mathscr{M}(x_U; \bar{\delta}, P_U(\bar{\delta})) \;=\; e^{-D_U},$$

where the second equality uses $\mathbb{E}[P_I \mid \bar{\delta}] = \mathbb{E}[\Pr(B = 1 \mid S, \xi) \mid \bar{\delta}] = \Pr(B = 1 \mid \bar{\delta})$ almost surely, by $\Pr(B = 1 \mid S, \xi) = P_I$ (shown in the proof of Lemma 5), the tower property, and the fact that $\bar{\delta}$ is a function of $(S, \xi)$; and the third equality holds because $x_U$ minimizes $\mathscr{M}(\cdot; \bar{\delta}, P_U(\bar{\delta}))$. Hence $\mathbb{E}[e^{-D_I} \mid \bar{\delta}] \le e^{-D_U}$ almost surely, and integrating, $\mathbb{E}[e^{-D_I}] \le \mathbb{E}[e^{-D_U}]$, that is, $G_I \ge G_U$.

If $P_I = P_U(\bar{\delta})$ almost surely, then $x(P_I, \bar{\delta}) = x_U$ and $D_I = D_U$ almost surely, so $\Delta_G = 0$. Conversely, if $\Delta_G = 0$ then $\mathbb{E}[e^{-D_I}] = \mathbb{E}[\mathscr{M}(x_U; \bar{\delta}, P_I)]$, and since the integrands are ordered pointwise by (32), they are equal almost surely; the minimizer of the strictly convex $\mathscr{M}(\cdot; \bar{\delta}, P_I)$ being unique, $x_U = x(P_I, \bar{\delta})$ almost surely, and by (5) this is $\operatorname{logit} P_U(\bar{\delta}) = \operatorname{logit} P_I$, that is, $P_I = P_U(\bar{\delta})$ almost surely. Q.E.D.

## A.11 Proof of Proposition 3

In the equilibrium of Theorem 1, $\mathscr{F}(\bar{\delta}) = \mathscr{F}(\Xi_\vartheta)$ and $\operatorname{logit} P_U(\bar{\delta}) = \eta_0 + \beta_\vartheta \Xi_\vartheta$; for $\vartheta > 0$, $\beta_\vartheta > 0$ and $\ell := \beta_\vartheta \Xi_\vartheta$ generates the same information as the price. Conditional on $B$, $\ell$ is normal with mean $(2B-1)\bar{\rho}^2_{U,\vartheta}/2$ and variance $\bar{\rho}^2_{U,\vartheta}$ (Appendix A.6, fact (d), and (17)), which is the form of Assumption 1. At $\vartheta = 0$, $\Xi_0 = \xi$ is independent of $B$, so the price carries no information about the event, the degenerate Gaussian signal of precision $\bar{\rho}_{U,0} = 0$.

From $\bar{\rho}^2_{U,\vartheta} = \vartheta^2\tau^2\bar{\rho}^4/(\vartheta^2\tau^2\bar{\rho}^2 + \sigma^2_\xi)$, $\partial_\vartheta \bar{\rho}^2_{U,\vartheta} = 2\vartheta\tau^2\bar{\rho}^4\sigma^2_\xi/v^2_\vartheta > 0$ and $\partial_{\sigma_\xi}\bar{\rho}^2_{U,\vartheta} = -2\vartheta^2\tau^2\bar{\rho}^4\sigma_\xi/v^2_\vartheta < 0$ for $\vartheta > 0$, and $b$ does not appear.

For the garbling, let $\bar{\rho}_1 > \bar{\rho}_2 > 0$ and let $\ell_1$ have the conditional distributions $N((2B-1)\bar{\rho}^2_1/2, \bar{\rho}^2_1)$; then $\ell_2 := (\bar{\rho}^2_2/\bar{\rho}^2_1)\ell_1 + \epsilon$, with $\epsilon \sim N(0, \bar{\rho}^2_2(1-\bar{\rho}^2_2/\bar{\rho}^2_1))$ independent of $(B, \ell_1)$, has the conditional distributions $N((2B-1)\bar{\rho}^2_2/2, \bar{\rho}^2_2)$: the conditional mean is $(\bar{\rho}^2_2/\bar{\rho}^2_1)(2B-1)\bar{\rho}^2_1/2$ and the conditional variance $\bar{\rho}^4_2/\bar{\rho}^2_1 + \bar{\rho}^2_2 - \bar{\rho}^4_2/\bar{\rho}^2_1$. So the signal of precision $\bar{\rho}_2$ is obtained from the signal of precision $\bar{\rho}_1$ by a transformation that does not depend on $B$, which is a garbling; for $\bar{\rho}_2 = 0$ the constant $\ell_2 := 0$ is such a garbling. Mutual information respects garbling by the data-processing inequality. Q.E.D.

## A.12 Proof of Theorem 2

Throughout, $\eta_0 := \operatorname{logit}\delta_0$, $u$ denotes a value of $\Xi_\vartheta$, $p(u) := \bar{\delta}$ is the execution price in the state $\Xi_\vartheta = u$ (a function of $u$ alone by (12)), and $m(u) := \sigma(\eta_0 + \beta_\vartheta u) = \Pr(B = 1 \mid \Xi_\vartheta = u)$ by (29); in the equilibrium of Theorem 1, $P_U(\bar{\delta}) = m(\Xi_\vartheta)$ in every state, so $D_U = \mathrm{KL}(p(u) \,\|\, m(u))$. Write $\bar{\rho}_I := \bar{\rho}_{I,\vartheta}$.

(i) We compute $\mathbb{E}[e^{-D_I} \mid \Xi_\vartheta = u]$. Conditional on $(\Xi_\vartheta = u, B)$, $L$ is normal with mean $\mu_{L|B}(u) := \beta_\vartheta u + (2B-1)\bar{\rho}^2_I/2$ and variance $\bar{\rho}^2_I$, by Lemma 6(i) and $\bar{\rho}^2\sigma^2_\xi/v_\vartheta = \bar{\rho}^2_I$, $(\bar{\rho}^2/2)(1 - \vartheta\tau\beta_\vartheta) = \bar{\rho}^2\sigma^2_\xi/(2v_\vartheta) = \bar{\rho}^2_I/2$ (using $1 - \vartheta\tau\beta_\vartheta = \sigma^2_\xi/v_\vartheta$, shown in Appendix A.9). Write $\phi_B(l \mid u)$ for that density and $f(l \mid u) := m(u)\phi_1(l \mid u) + (1 - m(u))\phi_0(l \mid u)$ for the conditional density of $L$ given $\Xi_\vartheta = u$. Since $\xi$ is independent of $(B, L)$, $\Pr(B = 1 \mid L = l, \Xi_\vartheta = u) = \Pr(B = 1 \mid L = l) = \sigma(\eta_0 + l) =: \theta(l)$ by (1), and Bayes' rule gives

$$m(u)\,\phi_1(l \mid u) \;=\; \theta(l)\,f(l \mid u), \qquad (1 - m(u))\,\phi_0(l \mid u) \;=\; (1 - \theta(l))\,f(l \mid u). \tag{33}$$

For $p, \theta \in (0,1)$, expanding the definition of $\mathrm{KL}(p \,\|\, \theta)$ gives $e^{-\mathrm{KL}(p \,\|\, \theta)} = (\theta/p)^p((1-\theta)/(1-p))^{1-p}$. Hence, with $p = p(u)$ and $D_I = \mathrm{KL}(p \,\|\, \theta(L))$,

$$\begin{aligned}
\mathbb{E}[e^{-D_I} \mid \Xi_\vartheta = u] &= \int \Big(\frac{\theta(l)f(l \mid u)}{p}\Big)^p \Big(\frac{(1-\theta(l))f(l \mid u)}{1-p}\Big)^{1-p} dl \\
&= \int \Big(\frac{m(u)\phi_1(l \mid u)}{p}\Big)^p \Big(\frac{(1-m(u))\phi_0(l \mid u)}{1-p}\Big)^{1-p} dl \\
&= e^{-\mathrm{KL}(p \,\|\, m(u))} \int \phi_1(l \mid u)^p\, \phi_0(l \mid u)^{1-p}\, dl,
\end{aligned}$$

where the first equality writes $f^p f^{1-p} = f$ and the second uses (33). For two normal densities with means $m_1, m_0$ and common variance $s^2$, the identity $p(l-m_1)^2 + (1-p)(l-m_0)^2 = (l-\bar{m})^2 + p(1-p)(m_1-m_0)^2$ with $\bar{m} := pm_1 + (1-p)m_0$ (expand both sides) gives $\phi_1^p \phi_0^{1-p}(l) = e^{-p(1-p)(m_1-m_0)^2/(2s^2)}\, \phi_{\bar{m},s^2}(l)$, with $\phi_{\bar{m},s^2}$ the normal density of mean $\bar{m}$ and variance $s^2$; integrating over $l$, $\int \phi_1^p \phi_0^{1-p}\, dl = e^{-p(1-p)(m_1-m_0)^2/(2s^2)}$. Here $m_1 - m_0 = \bar{\rho}_I^2$ and $s^2 = \bar{\rho}_I^2$, so the integral equals $\exp(-\frac{1}{2}\bar{\rho}_I^2\, p(1-p))$, and

$$\mathbb{E}[e^{-D_I} \mid \Xi_\vartheta = u] \;=\; e^{-D_U(u)} \exp(-\frac{1}{2}\bar{\rho}_I^2\, p(u)(1-p(u))). \tag{34}$$

Taking the expectation over $\Xi_\vartheta$ and substituting into (16) gives (18).

(ii) Since $0 < p(u) < 1$, the factor $p(1-p)$ lies in $(0, 1/4]$, so $\exp(-\frac{1}{2}\bar{\rho}_I^2 p(1-p)) \in [e^{-\bar{\rho}_I^2/8}, 1)$ in every state. By (18), $\Delta_G$ is $-\tau$ times the logarithm of a weighted average of this factor with the positive weights $e^{-D_U}$; the average is strictly below 1, so $\Delta_G > 0$. It is strictly above $e^{-\bar{\rho}_I^2/8}$ because $p(1-p) = 1/4$ only where $\bar{\delta} = 1/2$, an event of probability zero since $\bar{\delta}$ has a density (Proposition 1); hence $\Delta_G < \tau\bar{\rho}_I^2/8$.

(iii) Write $h := \operatorname{logit}\bar{\delta} - \eta_0$ and $\tilde{x} := \operatorname{logit}\delta^* - \eta_0$ in a state. By (12) and (9), $b\tilde{x} + \tau h = \varpi_\vartheta \Xi_\vartheta$, and $h$ has the sign of $\tilde{x}$ with $|h| < |\tilde{x}|$ (Lemma 1(iii)), so $|h| \le |\tilde{x}| \le \varpi_\vartheta |\Xi_\vartheta|/b$. Since $\tau\beta_\vartheta = \vartheta\tau^2\bar{\rho}^2/v_\vartheta \le \tau\bar{\rho}/(2\sigma_\xi)$ (maximize over $\vartheta$), $\varpi_\vartheta \le \bar{\varpi} := 1 + \tau\bar{\rho}/(2\sigma_\xi)$ for every $\vartheta$. As $\sigma' \le 1/4$, $|\bar{\delta} - \delta_0| \le |h|/4$, and as $|d(p(1-p))/dp| = |1-2p| \le 1$,

$$|\bar{\delta}(1-\bar{\delta}) - \delta_0(1-\delta_0)| \;\le\; \frac{\bar{\varpi}\,|\Xi_\vartheta|}{4b} \qquad \text{in every state.} \tag{35}$$

Write $\iota := \bar{\rho}_I^2/2 \le \bar{\rho}^2/2$ and $w_0 := \delta_0(1-\delta_0)$ inside this proof. Since $|e^{-\iota w} - e^{-\iota w_0}| \le \iota |w - w_0|$, (18) gives

$$\Big|\frac{\mathbb{E}[e^{-D_U} e^{-\iota\bar{\delta}(1-\bar{\delta})}]}{\mathbb{E}[e^{-D_U}]} - e^{-\iota w_0}\Big| \;\le\; \frac{\bar{\rho}^2}{2} \cdot \frac{\bar{\varpi}}{4b} \cdot \frac{\mathbb{E}[e^{-D_U}|\Xi_\vartheta|]}{\mathbb{E}[e^{-D_U}]} \;\le\; \frac{\bar{\rho}^2\bar{\varpi}}{8b} \cdot \frac{\mathbb{E}|\Xi_\vartheta|}{\mathbb{E}[e^{-D_U}]},$$

using $e^{-D_U} \le 1$ in the numerator. The numerator $\mathbb{E}|\Xi_\vartheta| \le \vartheta\tau\bar{\rho}^2/2 + \sqrt{v_\vartheta} \le \tau\bar{\rho}^2/2 + \sqrt{\tau^2\bar{\rho}^2 + \sigma_\xi^2}$ is bounded uniformly in $\vartheta$. For the denominator, Jensen's inequality and (27) give $\mathbb{E}[e^{-D_U}] \ge \exp(-\mathbb{E}[D_U]) \ge \exp(-\mathbb{E}[(h - \beta_\vartheta\Xi_\vartheta)^2]/8)$, and $\mathbb{E}[(h-\beta_\vartheta\Xi_\vartheta)^2] \le 2\mathbb{E}[h^2] + 2\mathbb{E}[(\beta_\vartheta\Xi_\vartheta)^2]$, where $\mathbb{E}[h^2] \le \bar{\varpi}^2\mathbb{E}[\Xi_\vartheta^2]/b^2$ and, conditional on $B$, $\beta_\vartheta\Xi_\vartheta$ is normal with mean $\pm\bar{\rho}_{U,\vartheta}^2/2$ and variance $\bar{\rho}_{U,\vartheta}^2 \le \bar{\rho}^2$ (Appendix A.6, fact (d), and (17)), so $\mathbb{E}[(\beta_\vartheta\Xi_\vartheta)^2] \le \bar{\rho}^4/4 + \bar{\rho}^2$. Hence $\mathbb{E}[e^{-D_U}] \ge c > 0$ for a constant $c$ that does not depend on $\vartheta$ or on $b \ge 1$. The weighted average in (18) therefore converges to $e^{-\iota w_0}$ uniformly in $\vartheta$ as $b \to \infty$; as it lies in $[e^{-\bar{\rho}^2/8}, 1]$, where the logarithm is Lipschitz, $\Delta_G \to \tau\iota w_0 = \tau\delta_0(1-\delta_0)\bar{\rho}_I^2/2$ uniformly in $\vartheta$. Q.E.D.

## A.13 Smoothness of the value of information

The proofs of Proposition 7, theorems 4 and 5, and corollary 1 rest on the following regularity of the value of information in the model's parameters. Throughout, $\eta_0 := \operatorname{logit}\delta_0$, $g$ is the map of Appendix A.1 from the terminal log-odds to the execution price, and

$$r(x) \;:=\; \operatorname{logit} g(\eta_0 + x) - \eta_0, \qquad x \in \mathbb{R}, \tag{36}$$

is the log-odds displacement of the execution price as a function of the terminal quote's.

**Lemma 7.** *Under Assumption 1, let*

$$F(\vartheta, \varepsilon; \sigma_\xi, \bar\rho) \;:=\; \Delta_G(\vartheta, 1/\varepsilon) \quad \textit{for } \varepsilon > 0, \qquad F(\vartheta, 0; \sigma_\xi, \bar\rho) \;:=\; \frac{1}{2}\tau\delta_0(1-\delta_0)\bar\rho^2_{I,\vartheta}.$$

*Then $F$ is smooth on $[0,1] \times [0,\infty) \times (0,\infty)^2$: it has partial derivatives of every order on the interior of this set, and each extends continuously to the boundary. In particular $\Delta_G$ is smooth in $(\vartheta, b, \sigma_\xi, \bar\rho)$ on $[0,1] \times (0,\infty)^3$.*

*Proof. Step 1: properties of $r$.* $r$ is smooth, since $g$ is smooth with values in $(0,1)$ (Appendix A.1). $r(0) = 0$ and $r'(0) = 1/2$, by $g(\eta_0) = \delta_0$ and $(\operatorname{logit} g)'(\eta_0) = 1/2$ (Appendix A.9). For $x \neq 0$, $r(x)/x \in (0,1)$, since $\bar\delta$ lies strictly between $\delta_0$ and $\delta^*$ (Lemma 1(iii)) and logit is strictly increasing; hence $|r(x)| \leq |x|$, and $r'(x) = (\operatorname{logit} g)'(\eta_0 + x) > 0$ (Appendix A.6, fact (b)). Finally, every derivative of $r$ of positive order is bounded by a polynomial in $|x|$. To see this, write $g_k(x) := g^{(k)}(\eta_0 + x) = \int_0^1 s^k \sigma^{(k)}(\eta_0 + sx)\, ds$ (differentiating under the integral in Appendix A.1), which is bounded by $\sup|\sigma^{(k)}| < \infty$; and note that $r^{(k)}$ is a polynomial in $g_1, \ldots, g_k$, $1/g$, and $1/(1-g)$, since $r = \log g - \log(1-g)$. For $x \geq 0$, $g(\eta_0 + x) \geq \delta_0$, and $1 - g(\eta_0 + x) = \int_0^1 (1 - \sigma(\eta_0 + sx))\, ds \geq \int_0^{\min(1,x)/x} (1 - \sigma(\eta_0 + sx))\, ds \geq \min(1, 1/x)\,(1 - \sigma(\eta_0 + 1))$, using that $1 - \sigma$ is decreasing; so $1/(1 - g(\eta_0 + x)) \leq C(1 + x)$. The case $x \leq 0$ is symmetric with the roles of $g$ and $1 - g$ exchanged. Hence $1/g$ and $1/(1-g)$ are bounded by $C(1 + |x|)$, and the claim follows.

*Step 2: the displacement.* For $\varepsilon \geq 0$, $u \in \mathbb{R}$, and $\vartheta \in [0,1]$, define $\tilde x = \tilde x(\varepsilon, u, \vartheta)$ as the solution of

$$\tilde x + \varepsilon\tau\, r(\tilde x) \;=\; \varepsilon\, \varpi_\vartheta u. \tag{37}$$

With $b = 1/\varepsilon$ and $u = \Xi_\vartheta$, (37) is $\Psi_b(\eta_0 + \tilde x) = \varpi_\vartheta \Xi_\vartheta$ divided by $b$, so by (12) $\tilde x$ is the terminal log-odds displacement $\operatorname{logit}\delta^* - \eta_0$ in the state $\Xi_\vartheta = u$. The left side of (37) is smooth in $\tilde x$ with derivative $1 + \varepsilon\tau r'(\tilde x) \geq 1$, it has the sign of $\tilde x$ with absolute value at least $|\tilde x|$ (Step 1), and it tends to $\pm\infty$ with $\tilde x$; so the solution exists, is unique, and satisfies

$$|\tilde x| \leq\; \varepsilon\, \varpi_\vartheta\, |u|. \tag{38}$$

By the implicit function theorem applied to the smooth map $(\varepsilon, u, \vartheta, \sigma_\xi, \bar\rho, \tilde x) \mapsto \tilde x + \varepsilon\tau r(\tilde x) -$

$\varepsilon\varpi_\vartheta u$, whose derivative in $\tilde{x}$ is at least 1, $\tilde{x}$ is smooth on the interior of its domain, and its partial derivatives are obtained by implicit differentiation: for instance

$$\partial_\varepsilon \tilde{x} = \frac{\varpi_\vartheta u - \tau r(\tilde{x})}{1 + \varepsilon\tau r'(\tilde{x})}, \qquad \partial_\vartheta \tilde{x} = \frac{\varepsilon\, \partial_\vartheta \varpi_\vartheta\, u}{1 + \varepsilon\tau r'(\tilde{x})}.$$

By induction on the order, each partial derivative of $\tilde{x}$ is a polynomial in $\varepsilon$, $u$, $\tau$, $\varpi_\vartheta$ and its derivatives, and the derivatives of $r$ at $\tilde{x}$, divided by a power of $1 + \varepsilon\tau r'(\tilde{x}) \geq 1$; by Step 1 and (38), it is therefore bounded by a polynomial in $|u|$, uniformly for $(\varepsilon, \vartheta, \sigma_\xi, \bar{\rho})$ in a compact set, and extends continuously to $\varepsilon = 0$ and to $\vartheta \in \{0, 1\}$. (The coefficients $\beta_\vartheta$, $\varpi_\vartheta$, $v_\vartheta$, and $\bar{\rho}_{I,\vartheta}$ are smooth in $(\vartheta, \sigma_\xi, \bar{\rho})$ on $[0,1] \times (0,\infty)^2$ because $v_\vartheta \geq \sigma_\xi^2 > 0$.) At $\varepsilon = 0$, $\tilde{x} = 0$ and $\partial_\varepsilon \tilde{x} = \varpi_\vartheta u$.

*Step 3: the integrands.* Conditional on $B$, write $\Xi_\vartheta = \mu_B + \sqrt{v_\vartheta}\, z$ with $z$ standard normal, where $\mu_B = (2B-1)\vartheta\tau\bar{\rho}^2/2$ (Appendix A.6, fact (d)). In the state $u = \mu_B + \sqrt{v_\vartheta} z$, put

$$h := r(\tilde{x}), \qquad \bar{\delta} := \sigma(\eta_0 + h), \qquad w := \bar{\delta}(1 - \bar{\delta}), \qquad a := \exp(-D(\eta_0 + \beta_\vartheta u,\, \eta_0 + h)),$$

with $D$ the Bregman divergence (26), so that $a = e^{-\mathrm{KL}\left(\bar{\delta} \,\|\, \sigma(\eta_0 + \beta_\vartheta u)\right)} = e^{-D_U}$ by (26) and (29). By (18), for $\varepsilon > 0$,

$$F \;=\; -\tau \log N + \tau \log Z, \qquad N := \sum_B \Pr(B)\, \mathbb{E}_z[a\, e^{-\bar{\rho}_{I,\vartheta}^2 w/2}], \qquad Z := \sum_B \Pr(B)\, \mathbb{E}_z[a], \quad (39)$$

where $\mathbb{E}_z$ integrates against the standard normal density. Each of $h$, $\bar{\delta}$, $w$, and $a$ is a smooth function of $(\varepsilon, \vartheta, \sigma_\xi, \bar{\rho}, z)$ by Step 2, and each of their partial derivatives of every order is bounded by a polynomial in $|z|$ uniformly for the parameters in a compact set: for $h$ and $\bar{\delta}$ this follows from Step 1, Step 2, and the boundedness of the derivatives of $\sigma$; for $a$, note that $0 < a \leq 1$ and that every partial derivative of $D$ of positive order is bounded by $C(1 + |\eta_1 - \eta_2|)$ for a constant $C$ depending on the order: from (26), $\partial_1 D(\eta_1, \eta_2) = \sigma(\eta_1) - \sigma(\eta_2)$ and $\partial_2 D(\eta_1, \eta_2) = -\sigma'(\eta_2)(\eta_1 - \eta_2)$, so by the Leibniz rule $\partial_2^k D = -\sigma^{(k)}(\eta_2)(\eta_1 - \eta_2) + (k-1)\,\sigma^{(k-1)}(\eta_2)$ for $k \geq 2$, while every derivative that involves $\partial_1$ at least once is a derivative of $\sigma$ evaluated at $\eta_1$ or at $\eta_2$, or zero; and all derivatives of $\sigma$ are bounded. Since $|\eta_1 - \eta_2| = |\beta_\vartheta u - h| \leq |\beta_\vartheta u| + |h|$ at the arguments of $a$, every derivative of $a$ is $a$ times a polynomial in quantities bounded by polynomials in $|u|$. A polynomial in $|z|$ is integrable against the standard normal density. Therefore, by the theorem on differentiation under the integral sign (if $f(\theta, z)$ is continuously differentiable in $\theta$ and $|\partial_\theta f(\theta, z)| \leq G(z)$ with $\int G\,\phi < \infty$ for $\theta$ in a neighborhood, then $\theta \mapsto \int f(\theta, z)\phi(z)dz$ is continuously differentiable with derivative $\int \partial_\theta f\, \phi$, by dominated convergence applied to difference quotients), $N$ and $Z$ are smooth in $(\vartheta, \varepsilon, \sigma_\xi, \bar{\rho})$ on the interior of the domain, with derivatives extending continuously to its boundary. Since $Z > 0$ and $N > 0$, $F$ is smooth as claimed. At $\varepsilon = 0$, Step 2 gives $\tilde{x} = 0$, hence $h = 0$, $\bar{\delta} = \delta_0$, and $w = \delta_0(1 - \delta_0)$ in every state, so $N = e^{-\bar{\rho}_{I,\vartheta}^2 \delta_0(1-\delta_0)/2} Z$ and (39)

equals $\tau\delta_0(1-\delta_0)\bar{\rho}^2_{I,\vartheta}/2$, which is the value assigned to $F(\vartheta,0)$; so $F$ is continuous at $\varepsilon=0$ and the extension is the one stated. Finally $\Delta_G(\vartheta,b)=F(\vartheta,1/b)$ is smooth in $b>0$ as a composition. Q.E.D.

Two consequences of the proof are used below. First, at $\varepsilon=0$ the weights $a$ reduce to $a_0(u):=\exp(-\mathrm{KL}(\delta_0\,\|\,\sigma(\eta_0+\beta_\vartheta u)))$, and the distribution of $\Xi_\vartheta$ reweighted by $a_0$ is normal:

$$\frac{\sum_B \Pr(B)\mathbb{E}_z[a_0\,X(u)]}{\sum_B \Pr(B)\mathbb{E}_z[a_0]} \;=\; \mathbb{E}[X(U)], \qquad U\sim N((2\delta_0-1)\vartheta\tau\bar{\rho}^2/2,\; v_\vartheta), \tag{40}$$

for every bounded or polynomially bounded $X$. Indeed, with $f_B$ the $N(\mu_B,v_\vartheta)$ density, $f:=\delta_0 f_1+(1-\delta_0)f_0$ the density of $\Xi_\vartheta$, and $m(u)=\sigma(\eta_0+\beta_\vartheta u)=\Pr(B=1\mid\Xi_\vartheta=u)$, Bayes' rule gives $mf=\delta_0 f_1$ and $(1-m)f=(1-\delta_0)f_0$, and $e^{-\mathrm{KL}(\delta_0\,\|\,m)}=(m/\delta_0)^{\delta_0}((1-m)/(1-\delta_0))^{1-\delta_0}$ as in Appendix A.12, so $a_0 f=f_1^{\delta_0}f_0^{1-\delta_0}=e^{-\delta_0(1-\delta_0)(\mu_1-\mu_0)^2/(2v_\vartheta)}\,\phi_{\bar{\mu},v_\vartheta}$ with $\bar{\mu}=\delta_0\mu_1+(1-\delta_0)\mu_0=(2\delta_0-1)\mu_1$, by the identity for $\phi_1^p\phi_0^{1-p}$ in Appendix A.12; the constant cancels in the ratio. Second, differentiating the expressions of Step 3 at $\varepsilon=0$, where $\tilde{x}=0$, $h=0$, $\bar{\delta}=\delta_0$, and $\partial_\varepsilon\tilde{x}=\varpi_\vartheta u$,

$$\begin{aligned} \partial_\varepsilon h=\frac{1}{2}\varpi_\vartheta u, \qquad \partial_\varepsilon\bar{\delta}=\frac{1}{2}\delta_0(1-\delta_0)\varpi_\vartheta u,\\ \partial_\varepsilon w=(1-2\delta_0)\,\partial_\varepsilon\bar{\delta}, \qquad \partial^2_\varepsilon w=(1-2\delta_0)\,\partial^2_\varepsilon\bar{\delta}-2(\partial_\varepsilon\bar{\delta})^2, \end{aligned} \tag{41}$$

by $r'(0)=1/2$, $\sigma'(\eta_0)=\delta_0(1-\delta_0)$, and the chain rule.

## A.14 The execution price in log-odds

The comparative statics of Section 3.3 rest on the map $r$ of (36), which sends the terminal quote's log-odds displacement to the execution price's, and on the secant ratio (19), which in the notation of (36) is $\varsigma(\delta^*)=r(x)/x$ at $x=\operatorname{logit}\delta^*-\eta_0$, with $\eta_0:=\operatorname{logit}\delta_0$. This lemma collects what is needed.

**Lemma 8.** *Let $\varsigma(x):=r(x)/x$ for $x\neq 0$ and $\varsigma(0):=1/2$.*

*(a) $r$ is a smooth, strictly increasing bijection of $\mathbb{R}$ onto $\mathbb{R}$ with $r(0)=0$, $r'(0)=1/2$, and $r''(0)=(1-2\delta_0)/12$; and $0<\varsigma(x)<1$ for $x\neq 0$.*

*(b) As $x\to+\infty$, $r(x)=\log x-\log(-\log\delta_0)-\eta_0+O(1/x)$; as $x\to-\infty$, $r(x)=-\log|x|+\log(-\log(1-\delta_0))-\eta_0+O(1/|x|)$. In particular $\varsigma(x)\to 0$ as $|x|\to\infty$, and there is $c>0$ with $\varsigma(x)\geq c\log(2+|x|)/(1+|x|)$ for every $x$.*

*(c) $\varsigma$ is continuous, and its supremum $\bar{\varsigma}(\delta_0)$ is attained and lies in $[1/2,1)$.*

*(d) If $\delta_0 = 1/2$, then $r$ is odd and $\varsigma(x) < 1/2$ for every $x \neq 0$, so $\bar{\varsigma}(1/2) = 1/2$. If $\delta_0 \neq 1/2$, then $\varsigma(x) = 1/2 + (1-2\delta_0)x/24 + O(x^2)$, so $\varsigma$ exceeds $1/2$ for small $x$ of the sign of $1-2\delta_0$, and $\bar{\varsigma}(\delta_0) > 1/2$.*

*Proof.* (a) Smoothness, $r(0) = 0$, $r'(0) = 1/2$, $r' > 0$, and $0 < r(x)/x < 1$ are Step 1 of the proof of Lemma 7. For $r''(0)$, write $g(\eta_0 + x) = \int_0^1 \sigma(\eta_0 + sx)\, ds$ (Appendix A.1), so that $g'(\eta_0) = \sigma'(\eta_0)/2 = w_0/2$ and $g''(\eta_0) = \sigma''(\eta_0)/3 = w_0(1-2\delta_0)/3$, with $w_0 := \delta_0(1-\delta_0)$, using $\sigma'' = \sigma'(1-2\sigma)$. Since $r(x) = \operatorname{logit} g(\eta_0 + x) - \eta_0$, $(\operatorname{logit})'(p) = 1/(p(1-p))$, and $(\operatorname{logit})''(p) = -(1-2p)/(p(1-p))^2$, the chain rule gives $r''(0) = -(1-2\delta_0)w_0^{-2}(w_0/2)^2 + w_0^{-1}w_0(1-2\delta_0)/3 = (1-2\delta_0)(1/3 - 1/4)$. Surjectivity follows from (b).

(b) Write $\mathscr{A}(\eta) = \eta + \log(1+e^{-\eta})$ and $\mathscr{A}(\eta_0) - \eta_0 = -\log\delta_0$. For $x > 0$, $1 - g(\eta_0 + x) = [x - \mathscr{A}(\eta_0 + x) + \mathscr{A}(\eta_0)]/x = [-\log\delta_0 - \log(1 + e^{-\eta_0 - x})]/x$, so $1 - g(\eta_0 + x) = (-\log\delta_0)/x + O(e^{-x}/x)$ and $g(\eta_0 + x) = 1 - O(1/x)$; hence $\operatorname{logit} g(\eta_0 + x) = \log g - \log(1-g) = \log x - \log(-\log\delta_0) + O(1/x)$. For $x < 0$, $\mathscr{A}(\eta_0 + x) = O(e^x)$ and $\mathscr{A}(\eta_0) = -\log(1-\delta_0)$, so $g(\eta_0 + x) = [\mathscr{A}(\eta_0) - \mathscr{A}(\eta_0 + x)]/|x| = (-\log(1-\delta_0))/|x| + O(e^x/|x|)$ and $\operatorname{logit} g(\eta_0 + x) = -\log|x| + \log(-\log(1-\delta_0)) + O(1/|x|)$. Subtracting $\eta_0$ gives the two expansions; they show $r(x) \to \pm\infty$ with $x$ and $\varsigma(x) \to 0$. The function $x \mapsto \varsigma(x)(1+|x|)/\log(2+|x|)$ is continuous and positive on $\mathbb{R}$ (at 0 it equals $1/(2\log 2)$; positivity elsewhere is (a)), and tends to 1 at $\pm\infty$ by the expansions, so it is bounded below by a positive $c$.

(c) $\varsigma$ is continuous on $\mathbb{R} \setminus \{0\}$, and at 0 because $r'(0) = 1/2$. It is positive, tends to 0 at $\pm\infty$, and equals $1/2$ at 0, so it attains its supremum, which is at least $1/2$ and, by (a), below 1.

(d) If $\delta_0 = 1/2$, then $\eta_0 = 0$ and $g(-x) = \int_0^1 \sigma(-sx)\, ds = \int_0^1 (1 - \sigma(sx))\, ds = 1 - g(x)$, so $\operatorname{logit} g(-x) = -\operatorname{logit} g(x)$ and $r$ is odd. For $x > 0$, $\sigma$ is strictly concave on $[0, \infty)$, so by Jensen's inequality $g(x) = \int_0^1 \sigma(sx)\, ds < \sigma(\int_0^1 sx\, ds) = \sigma(x/2)$, that is, $r(x) < x/2$; for $x < 0$ the same argument with the strict convexity of $\sigma$ on $(-\infty, 0]$ gives $r(x) > x/2$, and in both cases $\varsigma(x) < 1/2$. If $\delta_0 \neq 1/2$, Taylor's theorem with (a) gives $r(x) = x/2 + (1-2\delta_0)x^2/24 + O(x^3)$, hence the expansion of $\varsigma$ and the last two claims. Q.E.D.

**Lemma 9** (Displacement bounds)**.** *Under Assumption 1, fix $\vartheta \in [0,1]$, $b > 0$, and a state with $\Xi_\vartheta = u$, and write $\tilde{x} := \operatorname{logit}\delta^* - \eta_0$ and $h := \operatorname{logit}\bar{\delta} - \eta_0 = r(\tilde{x})$. Then*

$$b\,\tilde{x} + \tau\, r(\tilde{x}) \;=\; \varpi_\vartheta\, u, \tag{42}$$

*$\tilde{x}$ and $h$ have the sign of $u$, and, with $\varsigma := \varsigma(\delta^*)$,*

$$h \;=\; \varsigma\,\tilde{x}, \qquad \frac{\varpi_\vartheta|u|}{b+\tau} \;\le\; |\tilde{x}| \le\; \frac{\varpi_\vartheta|u|}{b}, \qquad |h| \le\; \frac{\varpi_\vartheta|u|}{b+\tau}, \tag{43}$$

*with strict inequalities when $u \neq 0$.*

*Proof.* By (12) and (9), $b\tilde{x} + \tau r(\tilde{x}) = \varpi_\vartheta u$, and since $r(\tilde{x})$ has the sign of $\tilde{x}$ (Lemma 8(a)), the left side has the sign of $\tilde{x}$; so $\tilde{x}$ and $h$ have the sign of $u$. Substituting $r(\tilde{x}) = \varsigma\tilde{x}$ gives

$\tilde{x}\,(b+\tau\varsigma) = \varpi_\vartheta u$ and $h\,(b/\varsigma+\tau) = \varpi_\vartheta u$; since $0<\varsigma<1$ for $\tilde{x}\neq 0$ and $\varsigma(\delta_0)=1/2$ (Lemma 8(a)), $b<b+\tau\varsigma<b+\tau<b/\varsigma+\tau$, which gives the bounds. Q.E.D.

## A.15 Proof of Proposition 4

Fix $\vartheta$ and a state with $\Xi_\vartheta = u \neq 0$, and keep the notation of Lemma 9, which gives the signs in (i).

(i) Differentiating (42) in $b$ at fixed $u$ gives $\tilde{x} + (b+\tau r'(\tilde{x}))\,\partial_b\tilde{x} = 0$, that is,

$$\partial_b\tilde{x} \;=\; -\,\frac{\tilde{x}}{b+\tau r'(\tilde{x})}, \qquad \partial_b h \;=\; r'(\tilde{x})\,\partial_b\tilde{x}, \tag{44}$$

with $r'(\tilde{x}) = d\,\mathrm{logit}\,\bar{\delta}/d\,\mathrm{logit}\,\delta^* > 0$ (the proof of Lemma 8(a)). Both derivatives have the sign opposite to $u$, so $|\tilde{x}|$ and $|h|$ are strictly decreasing in $b$. As $\sigma$ is strictly increasing, $|\bar{\delta}-\delta_0| = |\sigma(\eta_0+h)-\sigma(\eta_0)|$ is strictly increasing in $|h|$ at fixed sign of $h$, hence strictly decreasing in $b$.

(ii) By (44), $\partial_b(b\tilde{x}) = \tilde{x} + b\,\partial_b\tilde{x} = \tilde{x}\,\tau r'(\tilde{x})/(b+\tau r'(\tilde{x}))$, which has the sign of $\tilde{x}$, so $|b\tilde{x}|$ is strictly increasing. As $b\to\infty$, $|\tilde{x}| \leq \varpi_\vartheta|u|/b \to 0$ ((38) with $\varepsilon = 1/b$), so $\tau r(\tilde{x})\to 0$ and $b\tilde{x} = \varpi_\vartheta u - \tau r(\tilde{x}) \to \varpi_\vartheta u$. As $b\to 0$: the two terms on the left of (42) have the same sign, so $|\tau r(\tilde{x})| \leq \varpi_\vartheta|u|$, which confines $\tilde{x}$ to the bounded set $r^{-1}([-\varpi_\vartheta|u|/\tau, \varpi_\vartheta|u|/\tau])$ (Lemma 8(b)); hence $b\tilde{x}\to 0$. Q.E.D.

## A.16 Proof of Proposition 5

Fix $\vartheta\in(0,1]$ and $b$, and keep the notation of Lemma 9. Since $\varpi_\vartheta - \tau\beta_\vartheta = 1-\vartheta\tau\beta_\vartheta = \sigma_\xi^2/v_\vartheta$ (Appendix A.9), (42) with $r(\tilde{x}) = \varsigma\tilde{x}$ gives $u = \tilde{x}(b+\tau\varsigma)/\varpi_\vartheta$ and

$$\beta_\vartheta u - h \;=\; \frac{\tilde{x}}{\varpi_\vartheta}[\beta_\vartheta b - \varsigma(\varpi_\vartheta - \tau\beta_\vartheta)] \;=\; \frac{\tilde{x}}{\varpi_\vartheta v_\vartheta}[\vartheta\tau\bar{\rho}^2 b - \varsigma\,\sigma_\xi^2], \tag{45}$$

using $\beta_\vartheta v_\vartheta = \vartheta\tau\bar{\rho}^2$.

(i) By (13), $x_U = \tau(\beta_\vartheta u - h)$, and $\tilde{x}$ has the sign of $u$; (45) gives the three cases.

(ii) If $\sigma_\xi^2 < \vartheta\tau\bar{\rho}^2 b/\bar{\varsigma}(\delta_0)$, then $\varsigma(\delta^*)\leq\bar{\varsigma}(\delta_0) < \vartheta\tau\bar{\rho}^2 b/\sigma_\xi^2$ in every state, and (i) applies. If the reverse strict inequality holds, let $x^\circ$ be a point at which $\varsigma$ attains its supremum (Lemma 8(c)); the state $u^\circ := (bx^\circ + \tau r(x^\circ))/\varpi_\vartheta$ has terminal displacement $x^\circ$ by (42), and in a neighborhood of it, by continuity of $\varsigma$, the inequality of (i) is reversed.

(iii) By (i), $x_U\equiv 0$ would require $\varsigma$ to be constant, which it is not ($\varsigma(0)=1/2$ and $\varsigma\to 0$). For the probability, write $k := \vartheta\tau\bar{\rho}^2 b/\sigma_\xi^2$, so that $x_U$ has the sign opposite to $u$ exactly where $\varsigma(\tilde{x}) > k$, and note that $k\to 0$ as $b\to 0$. Fix $A>0$. On $\{|\Xi_\vartheta|\leq A\}$ the displacement is confined to the bounded set $r^{-1}([-\varpi_\vartheta A/\tau, \varpi_\vartheta A/\tau])$ (as in the proof of Proposition 4(ii)),

on which $\varsigma$ is bounded below by a positive constant that does not depend on $b$; so $\varsigma(\tilde{x}) > k$ there for all small $b$, and $\Pr(|\Xi_\vartheta| \leq A) \to 1$ as $A \to \infty$. Q.E.D.

## A.17 Proof of Proposition 6

Conditional on $B$, $\Xi_\vartheta \sim N(\mu_B, v_\vartheta)$ with $\mu_B = (2B-1)\vartheta\tau\bar{\rho}^2/2$ (Appendix A.6, fact (d)), and $v_\vartheta = \vartheta^2\tau^2\bar{\rho}^2 + \sigma_\xi^2$ is strictly increasing in $\sigma_\xi$.

(i) For $\vartheta \in (0,1)$, $\varpi_\vartheta = 1 + (1-\vartheta)\tau\beta_\vartheta$ is strictly decreasing in $\sigma_\xi$, as $\beta_\vartheta = \vartheta\tau\bar{\rho}^2/v_\vartheta$ is ((11)). The map $y \mapsto \tilde{x}$ solving $b\tilde{x} + \tau r(\tilde{x}) = y$ is strictly increasing and vanishes at 0 (Appendix A.3), so $|\tilde{x}|$ is a strictly increasing function of $|y|$; with $y = \varpi_\vartheta u$ at fixed $u \neq 0$, $|\tilde{x}|$ is strictly decreasing in $\sigma_\xi$, and so is $|h| = |r(\tilde{x})|$ (Lemma 8(a)).

(ii) As $\sigma_\xi \to \infty$ at fixed $b$ and $\vartheta$, $\mu_B$ is fixed and $v_\vartheta/\sigma_\xi^2 \to 1$, so $\Xi_\vartheta/\sigma_\xi$ converges in distribution, given $B$, to a standard normal; in particular $|\Xi_\vartheta| \to \infty$ in probability and $\Pr(\Xi_\vartheta > 0 \mid B) \to 1/2$. By Lemma 9, $\tilde{x}$ and $h$ have the sign of $u$ and $|\tilde{x}| \geq \varpi_\vartheta|u|/(b+\tau) \geq |u|/(b+\tau)$, so $|\tilde{x}| \to \infty$ in probability, and then $|h| = |r(\tilde{x})| \to \infty$ in probability by Lemma 8(b). Hence, for $\epsilon \in (0, 1/2)$ and $K_\epsilon := \text{logit}(1-\epsilon) - \eta_0$, $\Pr(\Xi_\vartheta > 0,\ \bar{\delta} \leq 1-\epsilon \mid B) = \Pr(0 < h \leq K_\epsilon \mid B) \to 0$, and symmetrically $\Pr(\Xi_\vartheta < 0,\ \bar{\delta} \geq \epsilon \mid B) \to 0$, while $\Pr(\Xi_\vartheta = 0 \mid B) = 0$; so $\bar{\delta} - \mathbf{1}\{\Xi_\vartheta > 0\} \to 0$ in probability, and with $\Pr(\Xi_\vartheta > 0 \mid B) \to 1/2$ the execution price converges in distribution to the two-point distribution with mass $1/2$ at 0 and at 1. Finally $\bar{\rho}^2_{U,\vartheta} = \vartheta^2\tau^2\bar{\rho}^4/v_\vartheta \to 0$. Q.E.D.

## A.18 Proof of Theorem 3

Fix $\vartheta \in (0,1]$ and $b$, write $\sigma := \sigma_\xi$, and keep $\langle X \rangle := \sum_B \Pr(B)\mathbb{E}_z[X]$ for the state expectation in the parametrization $u = \mu_B + \sqrt{v_\vartheta}\, z$ of Lemma 7, Step 3, with $a = e^{-D_U}$, $w = \bar{\delta}(1-\bar{\delta})$, and $\iota := \bar{\rho}^2_{I,\vartheta}/2 = \bar{\rho}^2\sigma^2/(2v_\vartheta)$. By (18), $\Delta_G = -\tau \log(\langle a\, e^{-\iota w}\rangle / \langle a \rangle)$.

(i) As $\sigma \to 0$, $\Delta_G < \tau\bar{\rho}^2_{I,\vartheta}/8$ by Theorem 2(ii), and $\bar{\rho}^2_{I,\vartheta} = \bar{\rho}^2\sigma^2/v_\vartheta \leq \sigma^2/(\vartheta^2\tau^2) \to 0$ by (17); so $\Delta_G \to 0$. As $\sigma \to \infty$, $\iota \leq \bar{\rho}^2/2$ is bounded, and $w \to 0$ in probability under the mixture distribution, since $\bar{\delta} - \mathbf{1}\{\Xi_\vartheta > 0\} \to 0$ in probability by Proposition 6(ii) and $w$ is uniformly continuous on $[0,1]$ and vanishes at 0 and at 1; so $\langle a(1 - e^{-\iota w})\rangle \leq \langle 1 - e^{-\iota w}\rangle \to 0$ by bounded convergence. For the denominator, write $m := \sigma(\eta_0 + \beta_\vartheta \Xi_\vartheta)$, so that $a = e^{-\text{KL}(\bar{\delta} \,\|\, m)} = (m/\bar{\delta})^{\bar{\delta}}((1-m)/(1-\bar{\delta}))^{1-\bar{\delta}}$. Conditional on $B$, $\beta_\vartheta\Xi_\vartheta$ is normal with mean $(2B-1)\bar{\rho}^2_{U,\vartheta}/2$ and variance $\bar{\rho}^2_{U,\vartheta}$ (Appendix A.6, fact (d), with (11) and (17)), and $\bar{\rho}_{U,\vartheta} \to 0$ as $\sigma_\xi \to \infty$ (Proposition 6(ii)); so $\beta_\vartheta\Xi_\vartheta \to 0$ and $m \to \delta_0$ in probability. On the event $\{\Xi_\vartheta > 0\}$, $\bar{\delta} \to 1$ in probability (Proposition 6(ii)) and $a \to \delta_0$, because $(1-\bar{\delta})\log(1-\bar{\delta}) \to 0$ and $(1-\bar{\delta})\log(1-m) \to 0$; on $\{\Xi_\vartheta < 0\}$, symmetrically, $a \to 1 - \delta_0$. Hence $\langle a \rangle \to \delta_0/2 + (1-\delta_0)/2 = 1/2$ by bounded convergence, the ratio $\langle a e^{-\iota w}\rangle / \langle a \rangle$ tends to 1, and $\Delta_G \to 0$.

(ii) $\Delta_G > 0$ on $(0, \infty)$ by Theorem 2(ii) and is continuous in $\sigma$ there by Lemma 7; with the limits in (i) it extends to a continuous function on $[0, \infty]$ that vanishes at both ends and

is positive inside, so it attains its maximum at an interior point $\sigma^*$. As $\Delta_G \to 0$, there is $\sigma' > \sigma^*$ with $\Delta_G(\sigma') < \Delta_G(\sigma^*)$, and the mean value theorem gives a noise level between them at which $\partial_\sigma \Delta_G < 0$. Q.E.D.

## A.19 Proof of Proposition 7

(i) By Lemma 7, $\Delta_G(\cdot, b)$ is continuous on $[0,1]$. Let $f(\vartheta) := \Delta_G(\vartheta, b) - c_I$. If $f(0) \le 0$, then $\vartheta^* = 0$ with the equilibrium of Theorem 1 at $\vartheta = 0$ satisfies Definition 3; if $f(1) \ge 0$, then $\vartheta^* = 1$ does. Otherwise $f(0) > 0 > f(1)$, and by the intermediate value theorem $f$ has a root in $(0,1)$, which is an interior acquisition equilibrium.

(ii) At an interior acquisition equilibrium with $\partial_\vartheta \Delta_G(\vartheta^*, b) \neq 0$, the map $(\vartheta, b, \sigma_\xi, \bar\rho, c_I) \mapsto \Delta_G(\vartheta, b; \sigma_\xi, \bar\rho) - c_I$ is smooth by Lemma 7, with a nonzero partial derivative in $\vartheta$ at the root, so by the implicit function theorem the root is locally a smooth function $\vartheta^*(b, \sigma_\xi, \bar\rho, c_I)$, and differentiating the identity $\Delta_G(\vartheta^*, b; \sigma_\xi, \bar\rho) = c_I$ in each parameter gives (21).

(iii) By Theorem 2(ii), $\Delta_G(\vartheta, b) < \tau \bar\rho_{I,\vartheta}^2/8 \le \tau\bar\rho^2/8 \le c_I$ for every $\vartheta \in [0,1]$, using $\bar\rho_{I,\vartheta} \le \bar\rho$ from (17). Hence $\Delta_G(0, b) < c_I$, so $\vartheta^* = 0$ is an acquisition equilibrium; no interior fraction satisfies $\Delta_G(\vartheta, b) = c_I$; and $\Delta_G(1, b) < c_I$ rules out $\vartheta^* = 1$. Q.E.D.

## A.20 Proof of Theorem 4

We first establish the following expansions, uniformly in $\vartheta \in [0,1]$ as $b \to \infty$:

$$\Delta_G(\vartheta, b) = \frac{\tau \delta_0(1-\delta_0)\bar\rho_{I,\vartheta}^2}{2} - \frac{\tau^2 \bar\rho_{I,\vartheta}^2\, \varpi_\vartheta\, \delta_0(1-\delta_0)(2\delta_0-1)^2\, \vartheta \bar\rho^2}{8b} + O(b^{-2}). \tag{46}$$

If $\delta_0 = 1/2$,

$$\Delta_G(\vartheta, b) = \frac{\tau \bar\rho_{I,\vartheta}^2}{8} - \frac{\tau \bar\rho_{I,\vartheta}^2\, \varpi_\vartheta^2\, \upsilon_\vartheta}{128 b^2} + O(b^{-3}). \tag{47}$$

Let $F$ be as in Lemma 7, with $\sigma_\xi$ and $\bar\rho$ fixed and suppressed, and write $F_\varepsilon$, $F_{\varepsilon\varepsilon}$, $F_\vartheta$ for its partial derivatives; write also $w_0 := \delta_0(1-\delta_0)$, $\iota_\vartheta := \bar\rho_{I,\vartheta}^2/2$, and $\langle X \rangle := \sum_B \Pr(B) \mathbb{E}_z[X]$ for the state expectation of Lemma 7, so that $N = \langle a\, e^{-\iota_\vartheta w} \rangle$ and $Z = \langle a \rangle$ in (39).

*Step 1: the first derivative at $\varepsilon = 0$.* From (39), $F_\varepsilon = -\tau(N_\varepsilon/N - Z_\varepsilon/Z)$ with $N_\varepsilon = \langle a_\varepsilon e^{-\iota_\vartheta w} \rangle - \iota_\vartheta \langle a\, w_\varepsilon e^{-\iota_\vartheta w} \rangle$ and $Z_\varepsilon = \langle a_\varepsilon \rangle$, the subscript denoting $\partial_\varepsilon$. At $\varepsilon = 0$, $w = w_0$ in every state, so $e^{-\iota_\vartheta w}$ is the constant $e^{-\iota_\vartheta w_0}$, $N = e^{-\iota_\vartheta w_0} Z$, and $N_\varepsilon = e^{-\iota_\vartheta w_0}(\langle a_\varepsilon \rangle - \iota_\vartheta \langle a w_\varepsilon \rangle)$; hence

$$F_\varepsilon(\vartheta, 0) \;=\; \tau \iota_\vartheta \frac{\langle a_0\, w_\varepsilon \rangle}{\langle a_0 \rangle} \;=\; \tau \iota_\vartheta (1-2\delta_0)\, \frac{1}{2} w_0 \varpi_\vartheta\, \mathbb{E}[U] \;=\; -\frac{\tau^2 \bar\rho_{I,\vartheta}^2\, \varpi_\vartheta\, w_0 (2\delta_0-1)^2\, \vartheta \bar\rho^2}{8}, \tag{48}$$

where the second equality substitutes $w_\varepsilon$ from (41) and applies (40) with $X(u) = u$, and the third uses $\mathbb{E}[U] = (2\delta_0 - 1)\vartheta\tau\bar\rho^2/2$ and $\iota_\vartheta = \bar\rho_{I,\vartheta}^2/2$.

*Step 2: the second derivative at the symmetric prior.* Let $\delta_0 = 1/2$, so that $w_0 = 1/4$ and, by (41), $w_\varepsilon = 0$ in every state at $\varepsilon = 0$ and $w_{\varepsilon\varepsilon} = -2(\partial_\varepsilon \bar{\delta})^2 = -\varpi_\vartheta^2 u^2/32$ there. Differentiating once more, with $\partial_\varepsilon e^{-\iota_\vartheta w} = -\iota_\vartheta w_\varepsilon e^{-\iota_\vartheta w}$ and $\partial_\varepsilon^2 e^{-\iota_\vartheta w} = (-\iota_\vartheta w_{\varepsilon\varepsilon} + \iota_\vartheta^2 w_\varepsilon^2) e^{-\iota_\vartheta w}$,

$$N_{\varepsilon\varepsilon} = \langle a_{\varepsilon\varepsilon} e^{-\iota_\vartheta w}\rangle - 2\iota_\vartheta \langle a_\varepsilon w_\varepsilon e^{-\iota_\vartheta w}\rangle + \langle a\,(-\iota_\vartheta w_{\varepsilon\varepsilon} + \iota_\vartheta^2 w_\varepsilon^2)\, e^{-\iota_\vartheta w}\rangle.$$

At $\varepsilon = 0$, $w_\varepsilon = 0$ removes the second term and the $w_\varepsilon^2$ term, so $N_\varepsilon = e^{-\iota_\vartheta w_0} Z_\varepsilon$ and $N_{\varepsilon\varepsilon} = e^{-\iota_\vartheta w_0}(Z_{\varepsilon\varepsilon} - \iota_\vartheta \langle a w_{\varepsilon\varepsilon}\rangle)$. Since $F_{\varepsilon\varepsilon} = -\tau[(N_{\varepsilon\varepsilon}/N - (N_\varepsilon/N)^2) - (Z_{\varepsilon\varepsilon}/Z - (Z_\varepsilon/Z)^2)]$, the terms in $Z_\varepsilon$ and $Z_{\varepsilon\varepsilon}$ cancel and

$$F_{\varepsilon\varepsilon}(\vartheta, 0) \;=\; \tau\iota_\vartheta \,\frac{\langle a_0\, w_{\varepsilon\varepsilon}\rangle}{\langle a_0\rangle} \;=\; -\tau\iota_\vartheta\,\frac{\varpi_\vartheta^2}{32}\,\mathbb{E}[U^2] \;=\; -\frac{\tau\,\bar{\rho}_{I,\vartheta}^2\,\varpi_\vartheta^2\, v_\vartheta}{64}, \tag{49}$$

by (40) with $X(u) = u^2$, since $U \sim N(0, v_\vartheta)$ when $\delta_0 = 1/2$.

*Step 3: The expansions.* By Lemma 7, $F$ and its $\varepsilon$-derivatives up to order three are continuous on $[0,1]\times[0,1]$, hence bounded there. Taylor's theorem in $\varepsilon$ with the Lagrange remainder gives, for $\varepsilon \in (0,1]$, $F(\vartheta,\varepsilon) = F(\vartheta,0) + \varepsilon F_\varepsilon(\vartheta,0) + R_2$ with $|R_2| \le \frac{1}{2}\varepsilon^2 \sup_{[0,1]^2}|F_{\varepsilon\varepsilon}|$, uniformly in $\vartheta$; with $\varepsilon = 1/b$, $F(\vartheta,0) = \tau w_0 \bar{\rho}_{I,\vartheta}^2/2$, and (48), this is (46). If $\delta_0 = 1/2$, then $F_\varepsilon(\vartheta,0) = 0$ by (48), and the third-order Taylor formula with (49) gives (47), with $|R_3| \le \frac{1}{6}\varepsilon^3 \sup_{[0,1]^2}|F_{\varepsilon\varepsilon\varepsilon}|$.

*Step 4: part (i).* Fix $\underline{\vartheta} \in (0,1]$ and write $K := [\underline{\vartheta}, 1]$, a compact interval. If $\delta_0 \neq 1/2$, (48) gives $F_\varepsilon(\vartheta,0) \le -c_1 < 0$ on $K$, with $c_1 := \min_K \tau^2 \bar{\rho}_{I,\vartheta}^2 \varpi_\vartheta w_0 (2\delta_0 - 1)^2 \vartheta \bar{\rho}^2/8 > 0$ because $\bar{\rho}_{I,\vartheta} \ge \bar{\rho}\sigma_\xi/\sqrt{\tau^2\bar{\rho}^2 + \sigma_\xi^2} > 0$, $\varpi_\vartheta \ge 1$, and $\vartheta \ge \underline{\vartheta}$. Since $F_\varepsilon$ is continuous on the compact $K \times [0,1]$, there is $\varepsilon_1 > 0$ with $F_\varepsilon < 0$ on $K \times [0, \varepsilon_1]$. If $\delta_0 = 1/2$, then $F_\varepsilon(\vartheta,0) = 0$ and, by Taylor's theorem applied to $F_\varepsilon$, $F_\varepsilon(\vartheta,\varepsilon) = \varepsilon F_{\varepsilon\varepsilon}(\vartheta,0) + R$ with $|R| \le \frac{1}{2}\varepsilon^2 \sup_{[0,1]^2}|F_{\varepsilon\varepsilon\varepsilon}|$; by (49), $F_{\varepsilon\varepsilon}(\vartheta,0) \le -c_2 < 0$ on all of $[0,1]$ (as $\varpi_\vartheta \ge 1$ and $v_\vartheta \ge \sigma_\xi^2$), so $F_\varepsilon < 0$ on $[0,1]\times(0,\varepsilon_1]$ for some $\varepsilon_1 > 0$. In either case, since $\partial_b \Delta_G(\vartheta, b) = -b^{-2} F_\varepsilon(\vartheta, 1/b)$, the first inequality of (i) holds for $b > 1/\varepsilon_1$, on $K$ in the first case and on $[0,1]$ in the second. For the second inequality, $F_\vartheta(\vartheta,0) = \tau w_0\, \partial_\vartheta(\bar{\rho}_{I,\vartheta}^2/2) = -\tau w_0\, \vartheta\tau^2 \bar{\rho}^4 \sigma_\xi^2/v_\vartheta^2 \le -c_3 < 0$ on $K$, by (17) and $\vartheta \ge \underline{\vartheta}$; by continuity of $F_\vartheta$ on $K \times [0,1]$ there is $\varepsilon_2 > 0$ with $F_\vartheta < 0$ on $K \times [0, \varepsilon_2]$, and $\partial_\vartheta \Delta_G(\vartheta, b) = F_\vartheta(\vartheta, 1/b) < 0$ for $b > 1/\varepsilon_2$. Take $b_{\underline{\vartheta}} := \max(1/\varepsilon_1, 1/\varepsilon_2)$.

*Step 5: part (ii).* For $b > b_{\underline{\vartheta}}$, $\partial_\vartheta \Delta_G(\cdot, b) < 0$ on the interval $[\underline{\vartheta}, 1]$ by (i), so $\Delta_G(\cdot, b)$ is strictly decreasing there by the mean value theorem, and $\Delta_G(\vartheta, b) = c_I$ has at most one root in $[\underline{\vartheta}, 1]$. At such a root in $[\underline{\vartheta}, 1)$, an interior acquisition equilibrium, $\partial_\vartheta \Delta_G < 0$ and $\partial_b \Delta_G > 0$ by (i), so (21) gives $d\vartheta^*/db > 0$ and $d\vartheta^*/dc_I < 0$. Q.E.D.

## A.21 Proof of Theorem 5

For $\varkappa \geq 0$, let

$$\bar{\rho}_U^2 := \frac{\bar{\rho}^2 \varkappa^2}{1+\varkappa^2}, \qquad \ell \mid B \sim N\left((2B-1)\bar{\rho}_U^2/2, \bar{\rho}_U^2\right),$$

and define $\bar{\delta}_\ell := \bar{\delta}(\sigma(\Psi_b^{-1}(\tau\ell)))$. We first establish the expression

$$\mathscr{V}_b(\varkappa) = -\tau \log \frac{\mathbb{E}\left[\exp\left(-\mathrm{KL}(\bar{\delta}_\ell \,\|\, \sigma(\operatorname{logit}\delta_0 + \ell))\right) \exp\left(-\frac{\bar{\rho}^2}{2(1+\varkappa^2)}\bar{\delta}_\ell(1-\bar{\delta}_\ell)\right)\right]}{\mathbb{E}\left[\exp\left(-\mathrm{KL}(\bar{\delta}_\ell \,\|\, \sigma(\operatorname{logit}\delta_0 + \ell))\right)\right]}. \tag{50}$$

Fix $\delta_0$, $\tau$, $\bar{\rho}$, and $b$, write $\sigma := \sigma_\xi$ and $\eta_0 := \operatorname{logit}\delta_0$, and let $r$ be the map (36).

*Step 1: the model along a ray.* For $\varkappa \geq 0$ and $\sigma > 0$ put $\vartheta := \varkappa\sigma/(\tau\bar{\rho})$, so that $\varkappa_\vartheta = \varkappa$. Then, by (11) and (17),

$$v_\vartheta = \sigma^2(1+\varkappa^2), \qquad \beta_\vartheta = \frac{\varkappa\bar{\rho}}{\sigma(1+\varkappa^2)}, \qquad \bar{\rho}_{I,\vartheta}^2 = \frac{\bar{\rho}^2}{1+\varkappa^2}, \qquad \bar{\rho}_{U,\vartheta}^2 = \frac{\bar{\rho}^2\varkappa^2}{1+\varkappa^2},$$

and conditional on $B$, with $z$ standard normal, $\Xi_\vartheta = \sigma\hat{z}$ where $\hat{z} := (2B-1)\varkappa\bar{\rho}/2 + \sqrt{1+\varkappa^2}\, z$. The public log-odds update is $\ell := \beta_\vartheta \Xi_\vartheta = \varkappa\bar{\rho}\hat{z}/(1+\varkappa^2)$, whose conditional distribution is $N((2B-1)\bar{\rho}_{U,\vartheta}^2/2, \bar{\rho}_{U,\vartheta}^2)$ and does not depend on $\sigma$; and

$$\varpi_\vartheta \Xi_\vartheta \;=\; \Xi_\vartheta + (1-\vartheta)\tau\beta_\vartheta\Xi_\vartheta \;=\; \tau\ell + \sigma\Big(\hat{z} - \frac{\varkappa}{\bar{\rho}}\,\ell\Big).$$

By (12), (9), and (18), in the state $(B, z)$ the terminal log-odds displacement $\tilde{x} := \operatorname{logit}\delta^* - \eta_0$ solves $b\tilde{x} + \tau r(\tilde{x}) = \tau\ell + \sigma(\hat{z} - \varkappa\ell/\bar{\rho})$, the execution price is $\bar{\delta} = \sigma(\eta_0 + r(\tilde{x}))$, the weight is $a = \exp(-D(\eta_0 + \ell, \eta_0 + r(\tilde{x})))$ with $D$ as in (26), and

$$\Delta_G\Big(\frac{\varkappa\sigma}{\tau\bar{\rho}}, b\Big) \;=\; -\tau \log \frac{\langle a\, e^{-\bar{\rho}^2 w/(2(1+\varkappa^2))}\rangle}{\langle a\rangle} \;=:\; \mathscr{G}(\varkappa, \sigma), \qquad w := \bar{\delta}(1-\bar{\delta}), \tag{51}$$

with $\langle X\rangle := \sum_B \Pr(B)\mathbb{E}_z[X]$. The right side of (51) is defined for every $\sigma \in \mathbb{R}$, and at $\sigma = 0$ the displacement solves $b\tilde{x} + \tau r(\tilde{x}) = \tau\ell$, that is, $\eta_0 + \tilde{x} = \Psi_b^{-1}(\tau\ell)$, so that $\bar{\delta} = \bar{\delta}_\ell$. Thus, $\mathscr{G}(\varkappa, 0)$ equals the right-hand side of (50).

*Step 2: smoothness of $\mathscr{G}$.* The map $(\varkappa, \sigma, z, \tilde{x}) \mapsto b\tilde{x} + \tau r(\tilde{x}) - \tau\ell - \sigma(\hat{z} - \varkappa\ell/\bar{\rho})$ is smooth with derivative $b + \tau r'(\tilde{x}) \geq b > 0$ in $\tilde{x}$, so $\tilde{x}$ is a smooth function of $(\varkappa, \sigma, z)$ on $[0,\infty)\times\mathbb{R}\times\mathbb{R}$ by the implicit function theorem, with $|\tilde{x}| \leq |\tau\ell + \sigma(\hat{z} - \varkappa\ell/\bar{\rho})|/b$ (the left side has the sign of $\tilde{x}$ and absolute value at least $b|\tilde{x}|$), a polynomial in $|z|$; as in Steps 2 and 3 of the proof of Lemma 7, every partial derivative of $\tilde{x}$, $\bar{\delta}$, $w$, and $a$ is bounded by a polynomial in $|z|$ uniformly for $(\varkappa, \sigma)$ in a compact set, using Step 1 of that proof for the derivatives of $r$. Differentiation under the integral sign then shows that $\mathscr{G}$ is smooth on $[0,\infty)\times\mathbb{R}$.

Continuity of $\mathcal{G}$ at $\sigma = 0$ gives $\mathcal{V}_b(\varkappa) = \mathcal{G}(\varkappa, 0)$, which by Step 1 establishes (50). Moreover, for every $\bar{\varkappa} > 0$ there is $C$ with

$$\sup_{0\le\varkappa\le\bar{\varkappa}} |\mathcal{G}(\varkappa,\sigma) - \mathcal{V}_b(\varkappa)| \le C\sigma, \qquad \sup_{0\le\varkappa\le\bar{\varkappa}} |\partial_\varkappa \mathcal{G}(\varkappa,\sigma) - \mathcal{V}_b'(\varkappa)| \le C\sigma, \qquad 0 < \sigma \le 1, \quad (52)$$

by the mean value theorem in $\sigma$ applied to $\mathcal{G}$ and to $\partial_\varkappa \mathcal{G}$.

*Step 3: part (i).* Write $\bar{\varkappa}_c := \sqrt{\tau\bar{\rho}^2/(8c_I) - 1}$. At $\varkappa = 0$, $\ell = 0$, so $\bar{\delta}_\ell = \delta_0$, $a = 1$, and $\mathcal{V}_b(0) = \tau\delta_0(1-\delta_0)\bar{\rho}^2/2$; as $\Delta_G(0,b) = \mathcal{G}(0,\sigma) \to \mathcal{V}_b(0)$ by (52), $\Delta_G(0,b) > c_I$ for all small $\sigma$. By Theorem 2(ii), $\Delta_G(1,b) < \tau\bar{\rho}_{I,1}^2/8 = \tau\bar{\rho}^2\sigma^2/(8(\tau^2\bar{\rho}^2 + \sigma^2)) \to 0$, so $\Delta_G(1,b) < c_I$ for all small $\sigma$, and Proposition 7(i) gives an interior acquisition equilibrium; the same two inequalities rule out both corners. At any interior acquisition equilibrium, $c_I = \Delta_G(\vartheta^*, b) < \tau\bar{\rho}_{I,\vartheta^*}^2/8 = \tau\bar{\rho}^2/(8(1+\varkappa_{\vartheta^*}^2))$ by Theorem 2(ii) and Step 1, so $\varkappa_{\vartheta^*} < \bar{\varkappa}_c$, which is $\vartheta^* < \bar{\varkappa}_c\sigma/(\tau\bar{\rho})$; here $\bar{\varkappa}_c$ is well defined because $c_I < \mathcal{V}_b(0) \le \tau\bar{\rho}^2/8$. If $\varkappa_{\vartheta^*} \to \varkappa^\circ$ along a sequence $\sigma \to 0$, then $\varkappa^\circ \le \bar{\varkappa}_c$ and, by (52) and the continuity of $\mathcal{V}_b$, $c_I = \mathcal{G}(\varkappa_{\vartheta^*}, \sigma) \to \mathcal{V}_b(\varkappa^\circ)$, so $\varkappa^\circ$ is a root of $\mathcal{V}_b(\varkappa) = c_I$.

*Step 4: The limiting acquisition equation and part (iii).* The value $\mathcal{V}_b(\varkappa)$ is given by (51) at $\sigma = 0$, where the displacement solves $\tilde{x} + \varepsilon\tau r(\tilde{x}) = \varepsilon\tau\ell$ with $\varepsilon := 1/b$. This is (37) with $\varpi_\vartheta u$ replaced by $\tau\ell$, and $\ell = (2B-1)\bar{\rho}_U(\varkappa)^2/2 + \bar{\rho}_U(\varkappa)z$ with $\bar{\rho}_U(\varkappa) := \bar{\rho}\varkappa/\sqrt{1+\varkappa^2}$ smooth in $\varkappa \ge 0$; the argument of Lemma 7 therefore applies verbatim and shows that $\mathcal{V}(\varkappa,\varepsilon) := \mathcal{V}_{1/\varepsilon}(\varkappa)$ is smooth on $[0,\infty)\times[0,\infty)$ with $\mathcal{V}(\varkappa,0) = \tau\delta_0(1-\delta_0)\bar{\rho}^2/(2(1+\varkappa^2))$, the value at $\varepsilon = 0$ being $\tau\delta_0(1-\delta_0)$ times the residual constant, as in Step 3 of that proof. Since $\partial_\varkappa \mathcal{V}(\varkappa,0) = -\tau\delta_0(1-\delta_0)\bar{\rho}^2\varkappa/(1+\varkappa^2)^2 < 0$ for $\varkappa > 0$, the equation $\mathcal{V}(\varkappa,0) = c_I$ has the unique root $\varkappa_\infty = \sqrt{\mathcal{V}_b(0)/c_I - 1} > 0$. Every root of $\mathcal{V}_b(\varkappa) = c_I$ satisfies $\varkappa \le \bar{\varkappa}_c$, since $\mathcal{V}_b(\varkappa) \le \tau\bar{\rho}^2/(8(1+\varkappa^2))$ by the argument of Theorem 2(ii) applied to (50) (with the weak inequality $w \le 1/4$). Fix $\eta \in (0, \varkappa_\infty)$. On the compact set $[0, \bar{\varkappa}_c + 1] \setminus (\varkappa_\infty - \eta, \varkappa_\infty + \eta)$, $|\mathcal{V}(\cdot,0) - c_I|$ is bounded below by a positive constant, and on $[\varkappa_\infty - \eta, \varkappa_\infty + \eta]$, $\partial_\varkappa \mathcal{V}(\cdot,0)$ is bounded above by a negative constant; by the uniform convergence of $\mathcal{V}(\cdot,\varepsilon)$ and $\partial_\varkappa \mathcal{V}(\cdot,\varepsilon)$ to $\mathcal{V}(\cdot,0)$ and its derivative on $[0,\bar{\varkappa}_c+1]$ as $\varepsilon \to 0$ (continuity of $\mathcal{V}$ and $\partial_\varkappa \mathcal{V}$ on a compact set), there is $\bar{\varepsilon} > 0$ such that for $\varepsilon < \bar{\varepsilon}$ every root of $\mathcal{V}(\cdot,\varepsilon) = c_I$ in $[0, \bar{\varkappa}_c + 1]$ lies in $(\varkappa_\infty - \eta, \varkappa_\infty + \eta)$, where $\mathcal{V}(\cdot,\varepsilon)$ is strictly decreasing, and the intermediate value theorem supplies one root there. Hence for $b > \bar{b} := 1/\bar{\varepsilon}$ the equation $\mathcal{V}_b(\varkappa) = c_I$ has exactly one root $\varkappa_b$, with $|\varkappa_b - \varkappa_\infty| < \eta$; as $\eta$ is arbitrary, $\varkappa_b \to \varkappa_\infty$. Moreover $\mathcal{V}_b'(\varkappa_b) < 0$.

*Step 5: part (ii).* Fix $b > \bar{b}$. By Step 4, $\mathcal{V}_b - c_I$ vanishes on $[0, \bar{\varkappa}_c + 1]$ only at $\varkappa_b$, where its derivative is negative. Repeating the argument of Step 4 with (52) in place of the convergence in $\varepsilon$: for a fixed $\eta' > 0$ small enough that $\mathcal{V}_b' < 0$ on $[\varkappa_b - \eta', \varkappa_b + \eta']$, there is $\bar{\sigma}(b) > 0$ such that for $\sigma < \bar{\sigma}(b)$ every root of $\mathcal{G}(\cdot,\sigma) = c_I$ in $[0, \bar{\varkappa}_c + 1]$ lies in $(\varkappa_b - \eta', \varkappa_b + \eta')$, where $\partial_\varkappa \mathcal{G}(\cdot,\sigma) < 0$, and exactly one root $\varkappa^*(\sigma)$ lies there. By Step 3, every acquisition equilibrium at such a $\sigma$ is interior with $\varkappa_{\vartheta^*} \le \bar{\varkappa}_c$, and $\vartheta^*$ is an interior acquisition equilibrium if and only if $\mathcal{G}(\varkappa_{\vartheta^*}, \sigma) = c_I$ by (51); so the acquisition equilibrium

is unique, $\vartheta^*(\sigma) = \sigma\varkappa^*(\sigma)/(\tau\bar\rho)$, and $\varkappa^*(\sigma) \to \varkappa_b$ as $\sigma \to 0$ (shrink $\eta'$). By the implicit function theorem applied to $\mathcal{G}$, which is smooth with $\partial_\varkappa \mathcal{G} \neq 0$ at $(\varkappa^*(\sigma), \sigma)$, $\varkappa^*$ is smooth on $(0, \bar\sigma(b))$ with $d\varkappa^*/d\sigma = -\partial_\sigma \mathcal{G}/\partial_\varkappa \mathcal{G}$, which is bounded on $(0, \bar\sigma(b))$ because $\partial_\sigma \mathcal{G}$ is bounded on compact sets and $\partial_\varkappa \mathcal{G}$ is bounded away from zero near $(\varkappa_b, 0)$. Hence $\vartheta^*$ is smooth in $\sigma$, $\vartheta^*/\sigma = \varkappa^*(\sigma)/(\tau\bar\rho) \to \varkappa_b/(\tau\bar\rho)$, and

$$\frac{d\vartheta^*}{d\sigma} \;=\; \frac{\varkappa^*(\sigma) + \sigma\, d\varkappa^*/d\sigma}{\tau\bar\rho} \;\longrightarrow\; \frac{\varkappa_b}{\tau\bar\rho} \;>\; 0 \qquad \text{as } \sigma \to 0,$$

so $d\vartheta^*/d\sigma > 0$ for all small $\sigma$, after shrinking $\bar\sigma(b)$. Q.E.D.

## A.22 Proof of Corollary 1

Let $F$ be as in Lemma 7, with $\sigma_\xi$ and $\bar\rho$ fixed and suppressed, and write $w_0 := \delta_0(1-\delta_0)$. By (17), $F(\vartheta, 0) = \tau w_0 \bar\rho^2 \sigma_\xi^2/(2v_\vartheta)$, which is continuous and strictly decreasing on $[0,1]$. The equation $F(\vartheta,0) = c_I$ reads $\vartheta^2\tau^2\bar\rho^2 = \sigma_\xi^2(\tau w_0\bar\rho^2/(2c_I) - 1) = \sigma_\xi^2\varkappa_\infty^2$, using $\mathcal{V}_b(0) = \tau w_0\bar\rho^2/2$ and the definition of $\varkappa_\infty$ in Theorem 5(iii); its unique root in $[0,1]$ is $\vartheta_\infty$, which lies in $(0,1)$ by the two hypotheses. Fix $\underline{\vartheta} \in (0, \vartheta_\infty)$, so that $F(\underline{\vartheta}, 0) > c_I > F(1,0)$.

Since $F$ is continuous on the compact $[0,1]\times[0,1]$ and $\Delta_G(\vartheta, b) = F(\vartheta, 1/b)$, $\Delta_G(\cdot, b) \to F(\cdot, 0)$ uniformly on $[0,1]$ as $b \to \infty$. Hence for all $b$ large enough, $\Delta_G(\vartheta, b) > c_I$ for every $\vartheta \in [0, \underline{\vartheta}]$, where $F(\vartheta,0) \geq F(\underline{\vartheta}, 0) > c_I$, and $\Delta_G(1, b) < c_I$. For $b > b_{\underline{\vartheta}}$, $\partial_\vartheta \Delta_G(\cdot, b) < 0$ on $[\underline{\vartheta}, 1]$ by Theorem 4(i), so $\Delta_G(\cdot, b)$ is strictly decreasing there by the mean value theorem. Let $b'$ be at least both thresholds and fix $b > b'$. Neither corner condition of Definition 3 holds, since $\Delta_G(0,b) > c_I$ and $\Delta_G(1,b) < c_I$. An interior acquisition equilibrium is a root of $\Delta_G(\cdot, b) = c_I$ in $(0,1)$; there is none in $(0, \underline{\vartheta}]$, and on $[\underline{\vartheta}, 1]$ the intermediate value theorem gives a root, which the strict decrease makes the only one. So the acquisition equilibrium is unique and interior. Any limit point $\vartheta^\circ$ of $\vartheta^*(b)$ as $b \to \infty$ satisfies $F(\vartheta^\circ, 0) = c_I$ by uniform convergence and the continuity of $F(\cdot, 0)$, so $\vartheta^\circ = \vartheta_\infty$, and compactness of $[0,1]$ gives $\vartheta^*(b) \to \vartheta_\infty$. At $\vartheta_\infty$, $F(\vartheta_\infty, 0) = \tau w_0 \bar\rho_{I,\vartheta_\infty}^2/2 = c_I$ gives the residual precision, and (17) the public one; $\vartheta_\infty = \varkappa_\infty \sigma_\xi/(\tau\bar\rho)$ is proportional to $\sigma_\xi$ with $\varkappa_\infty$ free of it. Q.E.D.

## A.23 Proof of Theorem 6

Let $F(\vartheta, \varepsilon; \sigma_\xi)$ be the function of Lemma 7 with $\bar\rho$ fixed and suppressed, so that $F(\vartheta, 0; \sigma_\xi) = \tau w_0 \bar\rho^2 \sigma_\xi^2/(2v_\vartheta)$ with $w_0 := \delta_0(1-\delta_0)$ by (17), and write $F_\sigma$ and $F_\vartheta$ for its partial derivatives in $\sigma_\xi$ and $\vartheta$. Differentiating $F(\vartheta, 0; \sigma_\xi)$, with $v_\vartheta - \vartheta^2\tau^2\bar\rho^2 = \sigma_\xi^2$,

$$F_\sigma(\vartheta, 0; \sigma_\xi) \;=\; \frac{\tau w_0\, \tau^2\bar\rho^4\, \vartheta^2 \sigma_\xi}{v_\vartheta^2}, \qquad F_\vartheta(\vartheta, 0; \sigma_\xi) \;=\; -\frac{\tau w_0\, \tau^2\bar\rho^4\, \vartheta \sigma_\xi^2}{v_\vartheta^2}.$$

On the compact $K \subset (0,1] \times (0,\infty)$ both are bounded away from zero, the first above and the second below, since $\vartheta$ and $\sigma_\xi$ are bounded away from zero there. By Lemma 7, $F_\sigma$ and $F_\vartheta$ are continuous on $K \times [0,1]$, so there is $\varepsilon_K > 0$ with $F_\sigma > 0$ and $F_\vartheta < 0$ on $K \times [0, \varepsilon_K]$. Since $\partial_{\sigma_\xi} \Delta_G = F_\sigma(\vartheta, 1/b; \sigma_\xi)$ and $\partial_\vartheta \Delta_G = F_\vartheta(\vartheta, 1/b; \sigma_\xi)$, both signs hold for $b > b_K := 1/\varepsilon_K$. At an interior acquisition equilibrium with $(\vartheta^*, \sigma_\xi) \in K$ and $b > b_K$, $\partial_\vartheta \Delta_G \neq 0$, and (21) gives $d\vartheta^*/d\sigma_\xi = -\partial_{\sigma_\xi} \Delta_G / \partial_\vartheta \Delta_G > 0$. Q.E.D.

## A.24 Proofs of Propositions 8 and 9

Keep the notation of Lemma 9: in the state $\Xi_\vartheta = u$, $\tilde{x} = \operatorname{logit} \delta^* - \eta_0$ solves (42), $h = r(\tilde{x})$, $\bar{\delta} = \sigma(\eta_0 + h)$, and $x = b\tilde{x}$; write $m := P_U(\bar{\delta}) = \sigma(\eta_0 + \beta_\vartheta u)$ and $w_0 := \delta_0(1-\delta_0)$.

*Proposition 8(i).* $x$ and $\bar{\delta}$ are functions of $\Xi_\vartheta$ (Theorem 1), and $\mathbb{E}[B \mid \Xi_\vartheta] = m$ by Definition 2(i) and $\mathscr{F}(\bar{\delta}) = \mathscr{F}(\Xi_\vartheta)$; so $\mathbb{E}[\mathscr{L} \mid \Xi_\vartheta] = x\,(\mathbb{E}[B \mid \Xi_\vartheta] - \bar{\delta}) = x(m - \bar{\delta})$, and the tower property gives $\mathbb{E}[\mathscr{L}] = \mathbb{E}[x(m - \bar{\delta})]$. By (13), $x_U = \tau(\operatorname{logit} m - \operatorname{logit} \bar{\delta})$ has the sign of $m - \bar{\delta}$, as logit is strictly increasing, so $x(m - \bar{\delta})$ has the sign of $x\, x_U$.

*Proposition 8(ii).* Multiplying the clearing identity $\vartheta x_I + (1-\vartheta) x_U + \xi = x$ of (13) by $B - \bar{\delta}$ and taking expectations gives the decomposition. For the informed term, $x_I$ and $\bar{\delta}$ are functions of $(L, \Xi_\vartheta)$, and $\mathbb{E}[B \mid L, \Xi_\vartheta] = \mathbb{E}[B \mid L, \xi] = \mathbb{E}[B \mid L] = P_I$ because $\xi$ is independent of $(B, L)$; so, by (13) and $\operatorname{logit} P_I = \eta_0 + L$,

$$\mathbb{E}[x_I(B - \bar{\delta})] \;=\; \mathbb{E}[x_I(P_I - \bar{\delta})] \;=\; \tau\, \mathbb{E}[(\operatorname{logit} P_I - \operatorname{logit} \bar{\delta})(P_I - \bar{\delta})] \;\geq\; 0,$$

with a strictly positive integrand unless $P_I = \bar{\delta}$, that is, unless $L = \operatorname{logit} \bar{\delta} - \eta_0$; conditional on $\Xi_\vartheta = u$ and $B$, $L$ is normal (Lemma 6(i)) and this event is null, so the expectation is strictly positive. For the uninformed term, $x_U$ and $\bar{\delta}$ are functions of $\Xi_\vartheta$, so by the tower property $\mathbb{E}[x_U(B - \bar{\delta})] = \mathbb{E}[x_U(m - \bar{\delta})] = \tau \mathbb{E}[(\operatorname{logit} m - \operatorname{logit} \bar{\delta})(m - \bar{\delta})] \geq 0$, strictly unless $m = \bar{\delta}$ almost surely; since $x_U$ is a continuous function of $u$ (Lemma 6(ii)) and $\Xi_\vartheta$ has full support, $m = \bar{\delta}$ almost surely would make $x_U$ vanish in every state, which Proposition 5(iii) excludes for $\vartheta > 0$, and for $\vartheta = 0$, $x_U = -\tau h \neq 0$ whenever $u \neq 0$. For the noise term, $\mathbb{E}[\xi B] = \mathbb{E}[\xi]\mathbb{E}[B] = 0$ by independence, so $\mathbb{E}[\xi(B - \bar{\delta})] = -\mathbb{E}[\xi\, \bar{\delta}(\Xi_\vartheta)]$, where $\bar{\delta}(\cdot)$ denotes the price as a function of the order flow. Conditional on $L$, $\xi$ is $N(0, \sigma_\xi^2)$ with density $\phi_\sigma$, and $\xi \phi_\sigma(\xi) = -\sigma_\xi^2 \phi_\sigma'(\xi)$; integrating by parts, with the boundary terms vanishing because $\bar{\delta}$ is bounded and $\phi_\sigma$ decays, $\mathbb{E}[\xi\, \bar{\delta}(\vartheta\tau L + \xi) \mid L] = \sigma_\xi^2\, \mathbb{E}[\bar{\delta}'(\vartheta\tau L + \xi) \mid L]$, where $\bar{\delta}' = d\bar{\delta}/d\Xi_\vartheta$. By the chain rule through (42), $d\tilde{x}/du = \varpi_\vartheta/(b + \tau r'(\tilde{x}))$, so $\bar{\delta}' = \bar{\delta}(1-\bar{\delta})\, r'(\tilde{x})\, \varpi_\vartheta/(b + \tau r'(\tilde{x})) > 0$ (Lemma 8(a)). Hence $\mathbb{E}[\xi(B - \bar{\delta})] = -\sigma_\xi^2 \mathbb{E}[\bar{\delta}'] < 0$.

*Proposition 8(iii).* Every $x \in \mathbb{R}$ is the total order of some state, since $x = b(\Psi_b^{-1}(\varpi_\vartheta \Xi_\vartheta) - \eta_0)$ and $\Psi_b^{-1}$ maps $\mathbb{R}$ onto $\mathbb{R}$. In the state $B = 1$, $\mathscr{L} = x - b[\mathscr{A}(\eta_0 + x/b) - \mathscr{A}(\eta_0)]$ has derivative $1 - \sigma(\eta_0 + x/b) > 0$ in $x$, and writing $\mathscr{A}(\eta) = \eta + \log(1 + e^{-\eta})$, $\mathscr{L} = b[\mathscr{A}(\eta_0) - \eta_0] - b \log(1 + e^{-\eta_0 - x/b}) \to b[\mathscr{A}(\eta_0) - \eta_0] = -b \log \delta_0$ as $x \to \infty$, since $\mathscr{A}(\eta_0) - \eta_0 = \log(1 + e^{-\eta_0}) = -\log \delta_0$. In the state $B = 0$, $\mathscr{L} = -b[\mathscr{A}(\eta_0 + x/b) - \mathscr{A}(\eta_0)]$ is strictly decreasing in $x$ and tends to

$b\mathscr{A}(\eta_0) = -b\log(1-\delta_0)$ as $x \to -\infty$. So the supremum over states is $b\max\{-\log\delta_0, -\log(1-\delta_0)\} = bM(\delta_0)$. Q.E.D.

*Proposition 9(i).* Fix $u \neq 0$. Both terms on the left of (42) have the sign of $\tilde{x}$ (Lemma 8(a)), so $\tau|r(\tilde{x})| \leq \varpi_\vartheta|u| = \tau|r(\tilde{x}^0)|$ and, $r$ being strictly increasing with $r(0) = 0$, $|\tilde{x}| \leq |\tilde{x}^0|$ for every $b$; and $b\tilde{x} \to 0$ as $b \to 0$, so $\tau r(\tilde{x}) \to \varpi_\vartheta u$ and $\tilde{x} \to \tilde{x}^0$ by continuity of $r^{-1}$. Hence $h \to \varpi_\vartheta u/\tau$ and $\bar{\delta} \to \bar{\delta}^0$. By part (i) of Proposition 8, $\mathbb{E}[\mathscr{L}]/b = \mathbb{E}[\tilde{x}(m - \bar{\delta})]$, whose integrand converges pointwise to $\tilde{x}^0(m - \bar{\delta}^0)$ and is bounded by $|\tilde{x}^0|$. This bound is integrable: by Lemma 8(b) there is $C_0$ with $r(x) \geq \log x - C_0$ for $x \geq 1$ and $r(x) \leq -\log|x| + C_0$ for $x \leq -1$, so $|\tilde{x}^0| \leq \max\{1, \exp(\varpi_\vartheta|u|/\tau + C_0)\}$, and $\mathbb{E}[e^{c|\Xi_\vartheta|}] < \infty$ for a Gaussian mixture. Dominated convergence gives $\mathbb{E}[\mathscr{L}]/b \to \mathbb{E}[\tilde{x}^0(m - \bar{\delta}^0)] = -c_1$. For the sign, (11) and $1 - \vartheta\tau\beta_\vartheta = \sigma_\xi^2/v_\vartheta$ (Appendix A.9) give $\operatorname{logit}\bar{\delta}^0 - \operatorname{logit} m = u(\varpi_\vartheta/\tau - \beta_\vartheta) = u(1 - \vartheta\tau\beta_\vartheta)/\tau = u\sigma_\xi^2/(\tau v_\vartheta)$, which has the sign of $u$; so $\bar{\delta}^0 - m$ has the sign of $u$, and so does $\tilde{x}^0$, because $r(\tilde{x}^0) = \varpi_\vartheta u/\tau$ and $r$ has the sign of its argument. The integrand of $c_1$ is therefore strictly positive for $u \neq 0$, and $c_1 > 0$. Since $\mathbb{E}[\mathscr{L}]/b \to -c_1 < 0$, $\mathbb{E}[\mathscr{L}] < 0$ for all small $b$.

*Proposition 9(ii).* Put $\varepsilon := 1/b$ and let $\tilde{x}(\varepsilon, u)$ be the solution of (37), which by Step 2 of Lemma 7 is smooth on $[0, \varepsilon_0] \times \mathbb{R}$ with $\tilde{x}(0, u) = 0$ and $\partial_\varepsilon\tilde{x}(0, u) = \varpi_\vartheta u$. Differentiating (37) twice in $\varepsilon$,

$$\partial_\varepsilon\tilde{x} + \tau r(\tilde{x}) + \varepsilon\tau r'(\tilde{x})\partial_\varepsilon\tilde{x} = \varpi_\vartheta u, \qquad \partial_\varepsilon^2\tilde{x} + 2\tau r'(\tilde{x})\partial_\varepsilon\tilde{x} + \varepsilon\tau[r''(\tilde{x})(\partial_\varepsilon\tilde{x})^2 + r'(\tilde{x})\partial_\varepsilon^2\tilde{x}] = 0,$$

so at $\varepsilon = 0$, where $\tilde{x} = 0$ and $r'(0) = 1/2$, $\partial_\varepsilon^2\tilde{x}(0, u) = -\tau\varpi_\vartheta u$. The total order is $x = b\tilde{x} = X(\varepsilon, u)$ with $X(\varepsilon, u) := \tilde{x}(\varepsilon, u)/\varepsilon = \int_0^1 \partial_\varepsilon\tilde{x}(s\varepsilon, u)\, ds$, which is smooth on $[0, \varepsilon_0] \times \mathbb{R}$ with $X(0, u) = \varpi_\vartheta u$ and $\partial_\varepsilon X(0, u) = \dfrac{1}{2}\partial_\varepsilon^2\tilde{x}(0, u) = -\tau\varpi_\vartheta u/2$; its derivatives are bounded by polynomials in $|u|$, uniformly for the parameters in a compact set, by the bounds of Step 2 of Lemma 7. The price $\bar{\delta} = \sigma(\eta_0 + r(\tilde{x}))$ is smooth in $\varepsilon$ with $\partial_\varepsilon\bar{\delta}(0, u) = w_0\varpi_\vartheta u/2$ ((41)), and $m$ does not depend on $\varepsilon$. Hence $\mathscr{M}(\varepsilon) := \mathbb{E}[X(\varepsilon, \Xi_\vartheta)(m - \bar{\delta}(\varepsilon, \Xi_\vartheta))]$, which equals $\mathbb{E}[\mathscr{L}]$ for $\varepsilon > 0$, is smooth on $[0, \varepsilon_0]$ by the differentiation-under-the-integral argument of Step 3 of Lemma 7, and Taylor's theorem gives $\mathscr{M}(\varepsilon) = \mathscr{M}(0) + \varepsilon\mathscr{M}'(0) + O(\varepsilon^2)$. By the tower property, $\mathbb{E}[\Xi_\vartheta(m - \delta_0)] = \mathbb{E}[\Xi_\vartheta(B - \delta_0)] = \operatorname{Cov}(\Xi_\vartheta, B)$, and with $\mu_B = (2B - 1)\vartheta\tau\bar{\rho}^2/2$, $\operatorname{Cov}(\Xi_\vartheta, B) = \delta_0\mu_1 - \delta_0[\delta_0\mu_1 + (1 - \delta_0)\mu_0] = w_0(\mu_1 - \mu_0) = w_0\vartheta\tau\bar{\rho}^2$; so $\mathscr{M}(0) = \varpi_\vartheta w_0\vartheta\tau\bar{\rho}^2$. Next, $\mathbb{E}[\Xi_\vartheta^2] = v_\vartheta + \delta_0\mu_1^2 + (1 - \delta_0)\mu_0^2 = v_\vartheta + \vartheta^2\tau^2\bar{\rho}^4/4$, and

$$\begin{aligned}\mathscr{M}'(0) &= \mathbb{E}[\partial_\varepsilon X(0, \Xi_\vartheta)(m - \delta_0)] - \mathbb{E}[X(0, \Xi_\vartheta)\,\partial_\varepsilon\bar{\delta}(0, \Xi_\vartheta)] \\ &= -\frac{\tau\varpi_\vartheta}{2}\, w_0\vartheta\tau\bar{\rho}^2 - \frac{w_0\varpi_\vartheta^2}{2}\Big(v_\vartheta + \frac{\vartheta^2\tau^2\bar{\rho}^4}{4}\Big),\end{aligned}$$

that is, $\mathscr{M}'(0) = -K_\infty(\vartheta)$ with

$$K_\infty(\vartheta) := \frac{w_0}{2}\Big[\vartheta\tau^2\varpi_\vartheta\bar{\rho}^2 + \varpi_\vartheta^2\Big(v_\vartheta + \frac{\vartheta^2\tau^2\bar{\rho}^4}{4}\Big)\Big] > 0,$$

where the sign holds for every $\vartheta \in [0,1]$ because $\varpi_\vartheta \geq 1$ and $v_\vartheta > 0$. Finally $\partial_b \mathbb{E}[\mathscr{L}] = -\varepsilon^2 \mathscr{M}'(\varepsilon)$ and $\mathscr{M}'(\varepsilon) = -K_\infty(\vartheta) + O(\varepsilon)$, so $\partial_b \mathbb{E}[\mathscr{L}] > 0$ for all large $b$.

*Proposition 9(iii).* The expansion in the proof of part (ii) holds with a remainder that is uniform in $\vartheta \in [0,1]$, because $\mathscr{M}$ and its $\varepsilon$-derivatives are continuous on the compact $[0,1] \times [0,\varepsilon_0]$ by the argument of Lemma 7; so $\sup_{\vartheta\in[0,1]} |\mathbb{E}[\mathscr{L}](\vartheta, b) - L_\infty(\vartheta)| \to 0$ as $b \to \infty$, where $L_\infty(\vartheta) := \varpi_\vartheta \vartheta \tau \bar{\rho}^2 w_0$ is continuous in $\vartheta$. By Corollary 1, $\vartheta^*(b) \to \vartheta_\infty$, so $\mathbb{E}[\mathscr{L}](\vartheta^*(b), b) \to L_\infty(\vartheta_\infty)$. At $\vartheta_\infty$, write $\varkappa := \varkappa_{\vartheta_\infty} = \varkappa_\infty$, so that $v_{\vartheta_\infty} = \sigma_\xi^2(1+\varkappa^2)$, $\tau\beta_{\vartheta_\infty} = \varkappa^2/(\vartheta_\infty(1+\varkappa^2))$, and by (22) and (17) $\bar{\rho}^2/(1+\varkappa^2) = \bar{\rho}^2_{I,\vartheta_\infty}$ and $\bar{\rho}^2\varkappa^2/(1+\varkappa^2) = \bar{\rho}^2_{U,\vartheta_\infty}$. Then

$$\begin{aligned} L_\infty(\vartheta_\infty) &= \vartheta_\infty \tau\bar{\rho}^2 w_0 + (1-\vartheta_\infty)\,\frac{\varkappa^2}{1+\varkappa^2}\,\tau\bar{\rho}^2 w_0 \\ &= \tau w_0 \bar{\rho}^2_{U,\vartheta_\infty} + \vartheta_\infty\, \tau w_0 \bar{\rho}^2_{I,\vartheta_\infty} \;=\; \tau w_0 \bar{\rho}^2 - 2c_I + 2c_I\vartheta_\infty, \end{aligned}$$

using $\bar{\rho}^2_U + \bar{\rho}^2_I = \bar{\rho}^2$ and $\tau w_0 \bar{\rho}^2_{I,\vartheta_\infty} = 2c_I$ from Corollary 1. This is the displayed limit. The same computation reads the three counterparty terms of Proposition 8(ii) at $b = \infty$, where $\bar{\delta} = \delta_0$: the informed term is $\tau\mathbb{E}[L(P_I - \delta_0)] = \tau \operatorname{Cov}(L, B) = \tau w_0 \bar{\rho}^2$, the uninformed term is $\tau\beta_\vartheta \mathbb{E}[\Xi_\vartheta(m - \delta_0)] = \tau\beta_\vartheta w_0 \vartheta \tau \bar{\rho}^2 = \tau w_0 \bar{\rho}^2_{U,\vartheta}$, and the noise term vanishes because $\bar{\delta}' \to 0$; at $\vartheta_\infty$ the informed term is $\tau w_0 \bar{\rho}^2_U + 2c_I$. Q.E.D.

## A.25 Proof of Proposition 10

Keep the notation of Appendix A.24.

*Part (i).* Since $\bar{\delta}$ is a function of $\Xi_\vartheta$ and $\mathbb{E}[B \mid \Xi_\vartheta] = m$, the tower property gives $\mathcal{S}(\bar{\delta}) = \mathbb{E}[m \log \bar{\delta} + (1-m)\log(1-\bar{\delta})]$ and $\mathcal{S}(m) = \mathbb{E}[m \log m + (1-m)\log(1-m)]$; subtracting, $\mathcal{S}(m) - \mathcal{S}(\bar{\delta}) = \mathbb{E}[m \log(m/\bar{\delta}) + (1-m)\log((1-m)/(1-\bar{\delta}))] = \mathbb{E}[\mathrm{KL}(m \,\|\, \bar{\delta})]$.

*Part (ii).* $\bar{\delta} = m$ in every state means $x_U = \tau(\operatorname{logit} m - \operatorname{logit} \bar{\delta}) = 0$ in every state. For $\vartheta > 0$ this is excluded by Proposition 5(iii). For $\vartheta = 0$, $m = \delta_0$, while $\bar{\delta}$ is a strictly increasing function of $\Xi_\vartheta$ with $\bar{\delta} = \delta_0$ at $\Xi_\vartheta = 0$ (Theorem 1(ii) and (12)), so $\bar{\delta} \neq \delta_0$ off $\{\Xi_\vartheta = 0\}$.

*Part (iii), large liquidity.* With $\varepsilon = 1/b$ and $h = h(\varepsilon, u)$ as in Step 3 of Lemma 7, $\log \bar{\delta} = -\mathscr{A}(-\eta_0 - h)$ and $\log(1-\bar{\delta}) = -\mathscr{A}(\eta_0 + h)$, which are smooth in $\varepsilon$ with derivatives bounded by polynomials in $|u|$, since the derivatives of $\mathscr{A}$ are bounded and those of $h$ are so bounded. At $\varepsilon = 0$, $h = 0$ and $\partial_\varepsilon h = \varpi_\vartheta u/2$ ((41)), so, as $d\log\sigma(\eta)/d\eta = 1 - \sigma(\eta)$ and $d\log(1-\sigma(\eta))/d\eta = -\sigma(\eta)$, $\partial_\varepsilon[B\log\bar{\delta} + (1-B)\log(1-\bar{\delta})] = [B(1-\delta_0) - (1-B)\delta_0]\varpi_\vartheta u/2 = (B-\delta_0)\varpi_\vartheta u/2$ at $\varepsilon = 0$. The differentiation-under-the-integral argument of Lemma 7 makes $\mathcal{S}(\bar{\delta})$ smooth in $\varepsilon$ on $[0,\varepsilon_0]$, and Taylor's theorem gives $\mathcal{S}(\bar{\delta}) = \mathcal{S}(\delta_0) + \varepsilon(\varpi_\vartheta/2)\mathbb{E}[(B-\delta_0)\Xi_\vartheta] + O(\varepsilon^2)$, where $\mathbb{E}[(B-\delta_0)\Xi_\vartheta] = \operatorname{Cov}(\Xi_\vartheta, B) = w_0\vartheta\tau\bar{\rho}^2$ as computed in Appendix A.24. Hence $d\mathcal{S}(\bar{\delta})/db = -\varepsilon^2\partial_\varepsilon \mathcal{S}(\bar{\delta}) = -\varpi_\vartheta w_0 \vartheta\tau\bar{\rho}^2/(2b^2) + O(b^{-3})$, which is negative for all large $b$ when $\vartheta > 0$.

*Part (iii), small liquidity.* By part (i), $\mathcal{S}(\bar{\delta}) = \mathcal{S}(m) - \mathbb{E}[\mathrm{KL}(m \,\|\, \bar{\delta})]$, and by (26), $\mathrm{KL}(m \,\|\, \bar{\delta}) = D(\eta_0 + h, \eta_0 + \beta_\vartheta u)$. As $b \to 0$, $h \to \varpi_\vartheta u/\tau$ pointwise (proof of Proposition 9(i)),

so the integrand converges to $\mathrm{KL}(m \,\|\, \bar{\delta}^0)$; it is dominated by $|h-\beta_\vartheta u| \le \varpi_\vartheta |u|/\tau + \beta_\vartheta |u|$, since the integrand of (26) is bounded by one in absolute value and $|h| \le |r(\tilde{x}^0)|$ by the monotonicity shown there. Dominated convergence gives $\mathcal{S}(\bar{\delta}) \to \mathcal{S}(m) - \mathbb{E}[\mathrm{KL}(m \,\|\, \bar{\delta}^0)] = \mathcal{S}(\bar{\delta}^0)$, and the subtracted term is strictly positive, because $\operatorname{logit} \bar{\delta}^0 - \operatorname{logit} m = u\sigma_\xi^2/(\tau v_\vartheta) \neq 0$ for $u \neq 0$. For the derivative, $\partial_b D(\eta_0 + h, \eta_0 + \beta_\vartheta u) = (\bar{\delta} - m)\, \partial_b h$ by $\partial_1 D(\eta_1, \eta_2) = \sigma(\eta_1) - \sigma(\eta_2)$, and by (44) and the chain rule $\partial_b h = r'(\tilde{x}) \partial_b \tilde{x} = -r'(\tilde{x})\tilde{x}/(b + \tau r'(\tilde{x}))$, so $|\partial_b D| \le |\tilde{x}|/\tau \le |\tilde{x}^0|/\tau$, an integrable bound; differentiating under the integral, $d\mathcal{S}(\bar{\delta})/db = -\mathbb{E}[(\bar{\delta} - m)\partial_b h]$, whose integrand converges to $(\bar{\delta}^0 - m)\tilde{x}^0/\tau$ as $b \to 0$, since $r'(\tilde{x})/(b + \tau r'(\tilde{x})) \to 1/\tau$. Hence $d\mathcal{S}(\bar{\delta})/db \to \mathbb{E}[(\bar{\delta}^0 - m)\tilde{x}^0]/\tau = c_1/\tau > 0$.

*Part (iii), the interior maximum.* $b \mapsto \mathcal{S}(\bar{\delta})$ is continuous on $(0, \infty)$, tends to $\mathcal{S}(\bar{\delta}^0)$ at 0 and to $\mathcal{S}(\delta_0)$ at $\infty$, is strictly increasing on some $(0, b_1]$ because its derivative tends to $c_1/\tau > 0$ at 0, and is strictly decreasing on some $[b_2, \infty)$. So its supremum over $(0, \infty)$ exceeds both limits, and it is attained on the compact interval $[b_1, b_2]$. Q.E.D.

## A.26 Proof of Proposition 11

Keep the notation of Appendices A.24 and A.25: $\varepsilon := 1/b$, $m := P_U(\bar{\delta})$, $\tilde{x}(\varepsilon, u)$ the solution of (37), $X(\varepsilon, u) := \tilde{x}(\varepsilon, u)/\varepsilon$ the total order, and $\mathscr{M}(\vartheta, \varepsilon) := \mathbb{E}[X(\varepsilon, \Xi_\vartheta)(m - \bar{\delta})]$, which equals $\mathbb{E}[\mathscr{L}]$ for $\varepsilon > 0$. Write also $G(\vartheta, \varepsilon) := \mathcal{S}(\bar{\delta})$ for the score of the execution price at the fraction $\vartheta$ and the liquidity $1/\varepsilon$, extended by $G(\vartheta, 0) := \mathcal{S}(\delta_0)$, and $\mathscr{M}(\vartheta, 0) := L_\infty(\vartheta)$. The proof of part (ii) uses the following extension of Lemma 7 to these two objects.

**Lemma 10.** *Under Assumption 1, $\mathscr{M}$ and $G$ are smooth on $[0,1] \times [0, \infty)$, with $\mathscr{M}(\vartheta, 0) = L_\infty(\vartheta)$, $\partial_\varepsilon \mathscr{M}(\vartheta, 0) = -K_\infty(\vartheta)$, $G(\vartheta, 0) = \mathcal{S}(\delta_0)$, and $\partial_\varepsilon G(\vartheta, 0) = L_\infty(\vartheta)/2$, where $K_\infty(\vartheta) > 0$ is the constant defined in the proof of Proposition 9(ii); in particular, $\mathcal{S}(\bar{\delta})$ and $\mathbb{E}[\mathscr{L}]$ are smooth in $(\vartheta, b)$ on $[0,1] \times (0, \infty)$.*

*Proof.* Write the state expectation over the primitives $(B, L, \xi)$, so that $u = \Xi_\vartheta = \vartheta\tau L + \xi$ and the distribution of $(B, L, \xi)$ does not depend on $\vartheta$. The integrands of $\mathscr{M}$ and $G$ are $X(\varepsilon, \vartheta\tau L + \xi)\,(m - \bar{\delta})$ and $B \log \bar{\delta} + (1 - B) \log(1 - \bar{\delta})$, with $\bar{\delta} = \sigma(\eta_0 + r(\tilde{x}))$ and $m = \sigma(\eta_0 + \beta_\vartheta u)$. By Step 2 of Lemma 7, $\tilde{x}$ is smooth in $(\varepsilon, u, \vartheta)$ on $[0, \infty) \times \mathbb{R} \times [0,1]$ with derivatives of every order bounded by polynomials in $|u|$, uniformly for $(\varepsilon, \vartheta, \sigma_\xi, \bar{\rho})$ in a compact set, and so is $X = \tilde{x}/\varepsilon = \int_0^1 \partial_\varepsilon \tilde{x}(s\varepsilon, u)\, ds$, since $\tilde{x}(0, u) = 0$; the coefficients $\varpi_\vartheta$ and $\beta_\vartheta$ are smooth in $\vartheta$ on $[0,1]$; $r$, $\sigma$, and $\log \sigma$ are smooth with derivatives of every order bounded by polynomials in their argument (Step 1 of Lemma 7 and $\mathscr{A}' = \sigma \in (0,1)$). Hence both integrands are smooth in $(\vartheta, \varepsilon)$ with every partial derivative bounded, uniformly on compact subsets of $[0,1] \times [0, \infty)$, by a polynomial in $(|L|, |\xi|)$, which is integrable against the Gaussian distribution of $(L, \xi)$, and the differentiation-under-the-integral argument of Step 3 of Lemma 7 gives smoothness of $\mathscr{M}$ and $G$ on $[0,1] \times [0, \infty)$; with $\varepsilon = 1/b$, this is smoothness of $\mathcal{S}(\bar{\delta})$ and $\mathbb{E}[\mathscr{L}]$ in $(\vartheta, b)$ on $[0,1] \times (0, \infty)$. The values and $\varepsilon$-derivatives at $\varepsilon = 0$ are those computed in the proofs of Proposition 9(ii) and Proposition 10(iii), where

$\mathscr{M}(\vartheta,0)=\mathbb{E}[\varpi_\vartheta\Xi_\vartheta(m-\delta_0)]=L_\infty(\vartheta)$, $\partial_\varepsilon\mathscr{M}(\vartheta,0)=-K_\infty(\vartheta)$ by the computation of $\mathscr{M}'(0)$ in the former, and $\partial_\varepsilon G(\vartheta,0)=(\varpi_\vartheta/2)\operatorname{Cov}(\Xi_\vartheta,B)=L_\infty(\vartheta)/2$. Q.E.D.

*Proposition 11(i).* In every state, $b\tilde{x}+\tau r(\tilde{x})=\varpi_\vartheta\Xi_\vartheta$ with $\tilde{x}$ and $r(\tilde{x})$ of the same sign, so $b|\tilde{x}|\leq\varpi_\vartheta|\Xi_\vartheta|$, that is, $|x|\leq\varpi_\vartheta|\Xi_\vartheta|$. Since $|B-\bar{\delta}|\leq 1$, $|\mathbb{E}[\mathscr{L}]|\leq\varpi_\vartheta\mathbb{E}|\Xi_\vartheta|$. Conditional on $B$, $\Xi_\vartheta$ is normal with mean $\mu_B$, $|\mu_B|=\vartheta\tau\bar{\rho}^2/2\leq\tau\bar{\rho}^2/2$, and variance $v_\vartheta\leq\tau^2\bar{\rho}^2+\sigma_\xi^2$, so $\mathbb{E}|\Xi_\vartheta|\leq|\mu_B|+\sqrt{v_\vartheta}\leq\tau\bar{\rho}^2/2+\sqrt{\tau^2\bar{\rho}^2+\sigma_\xi^2}$, using $\mathbb{E}|Z|\leq\sqrt{\mathbb{E}Z^2}$ for the centered part. With $\varpi_\vartheta\leq\bar{\varpi}=1+\tau\bar{\rho}/(2\sigma_\xi)$ as in the proof of Theorem 2(iii), the expected loss is bounded in absolute value by the second summand of the bracket defining $\bar{b}$, uniformly in $\vartheta\in[0,1]$ and $b>0$. The score of a forecast in $(0,1)$ is strictly negative, so $\lambda[\mathcal{S}(\bar{\delta})-\mathcal{S}(\delta_0)]<-\lambda\mathcal{S}(\delta_0)$. Hence $V(b)<-\lambda\mathcal{S}(\delta_0)+\bar{\varpi}(\tau\bar{\rho}^2/2+\sqrt{\tau^2\bar{\rho}^2+\sigma_\xi^2})-c_0b=c_0(\bar{b}-b)\leq 0$ for $b\geq\bar{b}$.

*Proposition 11(ii).* Let $F(\vartheta,\varepsilon)$ be as in Lemma 7, so that $F(\vartheta,0)=\dfrac{1}{2}\tau w_0\bar{\rho}_{I,\vartheta}^2$ and, by (46), $F(\vartheta,\varepsilon)=F(\vartheta,0)-D(\vartheta)\varepsilon+O(\varepsilon^2)$ uniformly in $\vartheta$, with $D(\vartheta)$ the coefficient of $1/b$ in (46). The function $F(\cdot,0)$ is strictly decreasing on $[0,1]$, with derivative $F_\vartheta(\vartheta,0)=-\tau w_0\bar{\rho}^2\sigma_\xi^2\vartheta\tau^2\bar{\rho}^2/v_\vartheta^2<0$ for $\vartheta>0$, by (17). Under the hypotheses of Corollary 1, for $b>b'$ the acquisition equilibrium $\vartheta^*(b)$ is unique and interior, converges to $\vartheta_\infty\in(0,1)$, and satisfies $F(\vartheta_\infty,0)=c_I$, the last by the display of that corollary.

*Step 1: the rate of convergence.* At the equilibrium, $c_I=F(\vartheta^*,\varepsilon)=F(\vartheta^*,0)-D(\vartheta^*)\varepsilon+O(\varepsilon^2)$, so $F(\vartheta^*,0)-F(\vartheta_\infty,0)=O(\varepsilon)$. Since $F_\vartheta(\cdot,0)$ is continuous and nonzero at $\vartheta_\infty$, it is bounded away from zero on a neighborhood of $\vartheta_\infty$, which $\vartheta^*(b)$ enters for all large $b$; the mean value theorem then gives $\vartheta^*(b)-\vartheta_\infty=O(\varepsilon)$. For the derivative, (21) gives $d\vartheta^*/db=-\partial_bF/\partial_\vartheta F$ with $\partial_bF=-\varepsilon^2F_\varepsilon(\vartheta^*,\varepsilon)$, and $F_\varepsilon$ is bounded on the compact $[0,1]\times[0,1]$ (Lemma 7), while $F_\vartheta(\vartheta^*,\varepsilon)\to F_\vartheta(\vartheta_\infty,0)\neq 0$ by continuity. Hence $d\vartheta^*/db=O(\varepsilon^2)$, and $d\vartheta^*/db>0$ for all large $b$ by Theorem 4(ii).

*Step 2: the score of the public posterior.* By the tower property, $\mathcal{S}(P_U(\bar{\delta}))=\mathbb{E}[m\log m+(1-m)\log(1-m)]$ and $\mathcal{S}(\delta_0)=\mathbb{E}[m\log\delta_0+(1-m)\log(1-\delta_0)]$, so $\mathcal{S}(P_U(\bar{\delta}))-\mathcal{S}(\delta_0)=\mathbb{E}[\mathrm{KL}(m\,\|\,\delta_0)]=\mathbb{E}[\log(\Pr(B\mid\Xi_\vartheta)/\Pr(B))]$, the mutual information between the event and the price. By Proposition 3 it is increasing in $\vartheta$, and by Step 1 $\vartheta^*$ is increasing in $b$ for all large $b$; hence $\mathcal{S}(P_U(\bar{\delta}))$ is increasing in $b$ along the equilibrium.

*Step 3: the score of the execution price.* By the chain rule, $d\mathcal{S}(\bar{\delta})/db=\partial_bG+\partial_\vartheta G\,d\vartheta^*/db$ with $G$ as in Lemma 10. First, $\partial_bG=-\varepsilon^2G_\varepsilon(\vartheta^*,\varepsilon)$, and Taylor's theorem in $\varepsilon$ at fixed $\vartheta$, with $G_{\varepsilon\varepsilon}$ bounded on the compact, gives $G_\varepsilon(\vartheta^*,\varepsilon)=G_\varepsilon(\vartheta^*,0)+O(\varepsilon)=L_\infty(\vartheta^*)/2+O(\varepsilon)=L_\infty(\vartheta_\infty)/2+O(\varepsilon)$, the last by Step 1 and the smoothness of $L_\infty$. Second, $G(\vartheta,0)=\mathcal{S}(\delta_0)$ does not depend on $\vartheta$, so $G_\vartheta(\vartheta,0)=0$, and with $G_{\vartheta\varepsilon}$ bounded, $G_\vartheta(\vartheta^*,\varepsilon)=O(\varepsilon)$; with $d\vartheta^*/db=O(\varepsilon^2)$ from Step 1, the second term is $O(\varepsilon^3)$. Together, $d\mathcal{S}(\bar{\delta})/db=-L_\infty(\vartheta_\infty)/(2b^2)+O(b^{-3})$, which is negative for all large $b$ since $L_\infty(\vartheta_\infty)>0$.

*Step 4: the expected loss.* Likewise $d\mathbb{E}[\mathscr{L}]/db=\partial_b\mathscr{M}+\partial_\vartheta\mathscr{M}\,d\vartheta^*/db$. First, $\partial_b\mathscr{M}=-\varepsilon^2\mathscr{M}_\varepsilon(\vartheta^*,\varepsilon)=\varepsilon^2K_\infty(\vartheta^*)+O(\varepsilon^3)$ by Lemma 10, which is positive for all large $b$ since $K_\infty>0$ and $K_\infty(\vartheta^*)\to K_\infty(\vartheta_\infty)$. Second, $\mathscr{M}_\vartheta(\vartheta^*,\varepsilon)\to\mathscr{M}_\vartheta(\vartheta_\infty,0)=L'_\infty(\vartheta_\infty)$ by continuity.

Write $k_\vartheta := \vartheta\tau\beta_\vartheta = \vartheta^2\tau^2\bar{\rho}^2/v_\vartheta \in [0,1)$, so that $\vartheta\varpi_\vartheta = \vartheta + (1-\vartheta)k_\vartheta$ and $L_\infty(\vartheta) = \tau\bar{\rho}^2 w_0[\vartheta + (1-\vartheta)k_\vartheta]$; then $L'_\infty(\vartheta) = \tau\bar{\rho}^2 w_0[1 - k_\vartheta + (1-\vartheta)k'_\vartheta]$ with $k'_\vartheta = 2\vartheta\tau^2\bar{\rho}^2\sigma_\xi^2/v_\vartheta^2 \geq 0$, so $L'_\infty > 0$ on $[0,1]$. With $d\vartheta^*/db > 0$ from Step 1, the second term is positive for all large $b$, and $d\mathbb{E}[\mathscr{L}]/db > 0$.

*Step 5: the value.* $V'(b) = \lambda\, d\mathcal{S}(\bar{\delta})/db - d\mathbb{E}[\mathscr{L}]/db - c_0 < 0 - 0 - c_0 < 0$ for all large $b$, by Steps 3 and 4.